2

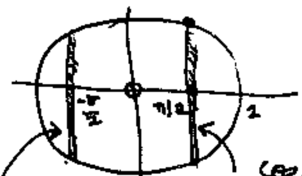
The Complex Function and its Derivative

sec 2.1

1] Note that the denominator of the given function $= 0$ if $z = \pm i$ or if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots$ or $x = \pm\left(n\pi - \frac{\pi}{2}\right)$ $n = 1, 2, 3 \ldots$

Thus $f(z)$ is defined everywhere in the domain $|z| < 1$. Thus $f(z)$ fails to be defined nowhere in $|z| < 1$

2] The points $z = \pm i$ lie in the domain $|z| < 1$. The function is not defined at $z = \pm i$

3] 

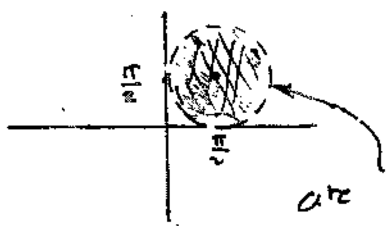
if $x = \pm\frac{\pi}{2}$ and $x^2 + y^2 = 4$

$y = \pm\sqrt{4 - \pi^2/4}$

$\cos x = 0$ here on this line

$\cos x = 0$ on this line. Thus in the domain $|z| < 2$, $f(z)$ is not defined at $z = \pm i$ or at any points on these two lines

$x = \pm\pi/2, \quad -\sqrt{4 - \pi^2/4} < y < \sqrt{4 - \pi^2/4}$

4] 

The domain $\left|z - (1+i)\frac{\pi}{2}\right| < \frac{\pi}{2}$ lies inside this circle. The points $\pm i$ are outside this domain.

$\cos x = 0$ if $x = \frac{\pi}{2}$. Points on the line $x = \frac{\pi}{2}$, $0 < y < \pi$ are in the domain and the function is undefined.

5] $f(1 + 2i) = (1+2i)^2 + 1 = 1 + 4i - 4 + 1$
$= -2 + 4i$

6] $z\bar{z} = 5 \quad \frac{1}{z\bar{z} - 5}$ is undefined

7.] $1+2i + \frac{1}{1+2i} + \text{Im}(1+2i)$

$= 1+2i + \frac{1-2i}{5} + 2 = 3\frac{1}{5} + i\frac{8}{5} =$

$= \frac{16+8i}{5}$

8] $\frac{1+i2}{\cos 1 + i\sin 2} = \frac{1+2i}{.5403 + i.9093}$

$= 2.1085 + i.1531$

9] $\frac{1}{z+i} = \frac{1}{x+iy+i} = \frac{1}{x+i(y+1)} =$

$\frac{x-i(y+1)}{x^2+(y+1)^2}$ $u = \frac{x}{x^2+(y+1)^2}$,

$v = \frac{-(y+1)}{x^2+(y+1)^2}$

10] $\frac{1}{z} + i = \frac{1}{x+iy} + i = \frac{x-iy}{x^2+y^2} + i$

$u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2} + 1$

11] $x+iy + \frac{1}{x+iy} = x+iy + \frac{(x-iy)}{x^2+y^2}$

$u = x + \frac{x}{x^2+y^2}$, $v = y - \frac{y}{x^2+y^2}$

12) $(x+iy)^3 + (x+iy) = x^3 + 3x^2 iy + 3x(-y^2) - iy^3 + x + iy$

$u = x^3 - 3xy^2 + x$

$v = 3x^2y - y^3 + y$

13] This is just the conjugate of problem 12, since $(\bar{z})^3 = \overline{z^3}$

$\therefore u = x^3 - 3xy^2 + x$, $v = -3x^2y + y^3 - y$

14)

$$x + i2y = \frac{1}{2}(z+\bar z) + i\frac{2}{2}\frac{(z-\bar z)}{i} =$$

$$\frac{z}{2} + z + \frac{\bar z}{2} - \bar z = \frac{3}{2}z - \frac{1}{2}\bar z$$

15)

$$\frac{2}{z+\bar z} + \frac{2i}{i(z-\bar z)} = 2\left[\frac{1}{z+\bar z} + \frac{1}{z-\bar z}\right]$$

$$= 2\,\frac{2z}{(z+\bar z)(z-\bar z)} = \frac{4z}{z^2-(\bar z)^2}$$

16)

$$i\,\frac{[z+\bar z]^2}{4} + \left(-\frac{1}{4}\right)\left(z-\bar z\right)^2$$

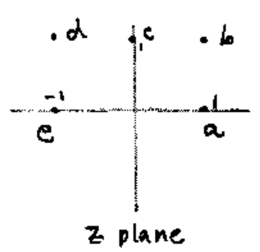
$$= \frac{-(z-\bar z)^2 + i(z+\bar z)^2}{4}$$

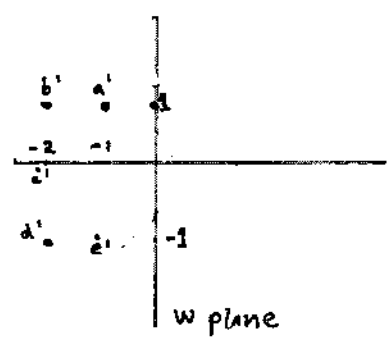
17) write as $x+iy + \dfrac{x+iy}{x^2+y^2} =$

$$z + \frac{z}{z\bar z} = z + 1/\bar z$$

18)

	z	$w = i(z+i)$
a	1	$-1+i = a'$
b	$1+i$	$-2+i = b'$
c	i	$-2 = c'$
d	$-1+i$	$-2-i = d'$
e	-1	$-1-i = e'$





19) $w = i/z$

z is same as previous problem

$w' = i/z$

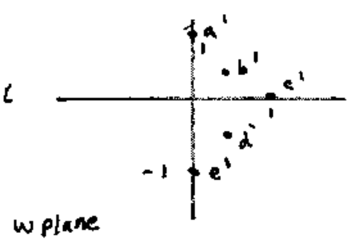
$a' = i$

$b' = \frac{1}{2} + \frac{1}{2}i$

$c' = 1$

$d' = 1/2 - 1/2\,i$

$e' = -i$

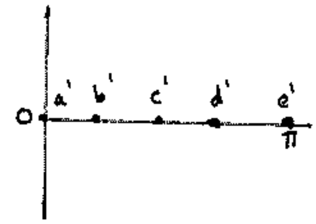


sec 2.1

20

	z
a	1
b	$1+i$
c	i
d	$-1+i$
e	-1

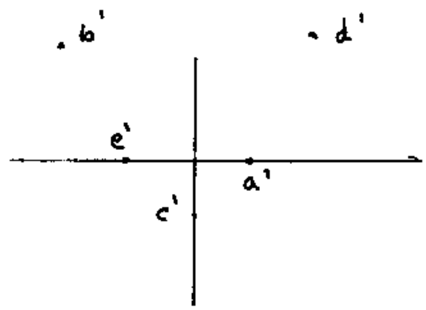
$w = \arg z$	
0	a'
$\pi/4$	b'
$\pi/2$	c'
$3\pi/4$	d'
π	e'



W plane

21) $w = z^3$. Values of same as 20.

$w = z^3$	
1	a'
$-2+2i$	b'
$-i$	c'
$2+2i$	d'
-1	e'



22) $f(z) = \frac{1}{z+i}$ $\quad f(1/z) = \frac{1}{\frac{1}{z}+i} = \frac{z}{iz+1}$

23) $f(f(z)) = \frac{1}{\left(\frac{1}{z+i}\right)+i} = \frac{z+i}{i(z+i)+1} = \frac{z+i}{iz}$

24) $f\left(\frac{1}{f(z)}\right) = ?$ $\quad \frac{1}{f(z)} = z+i$

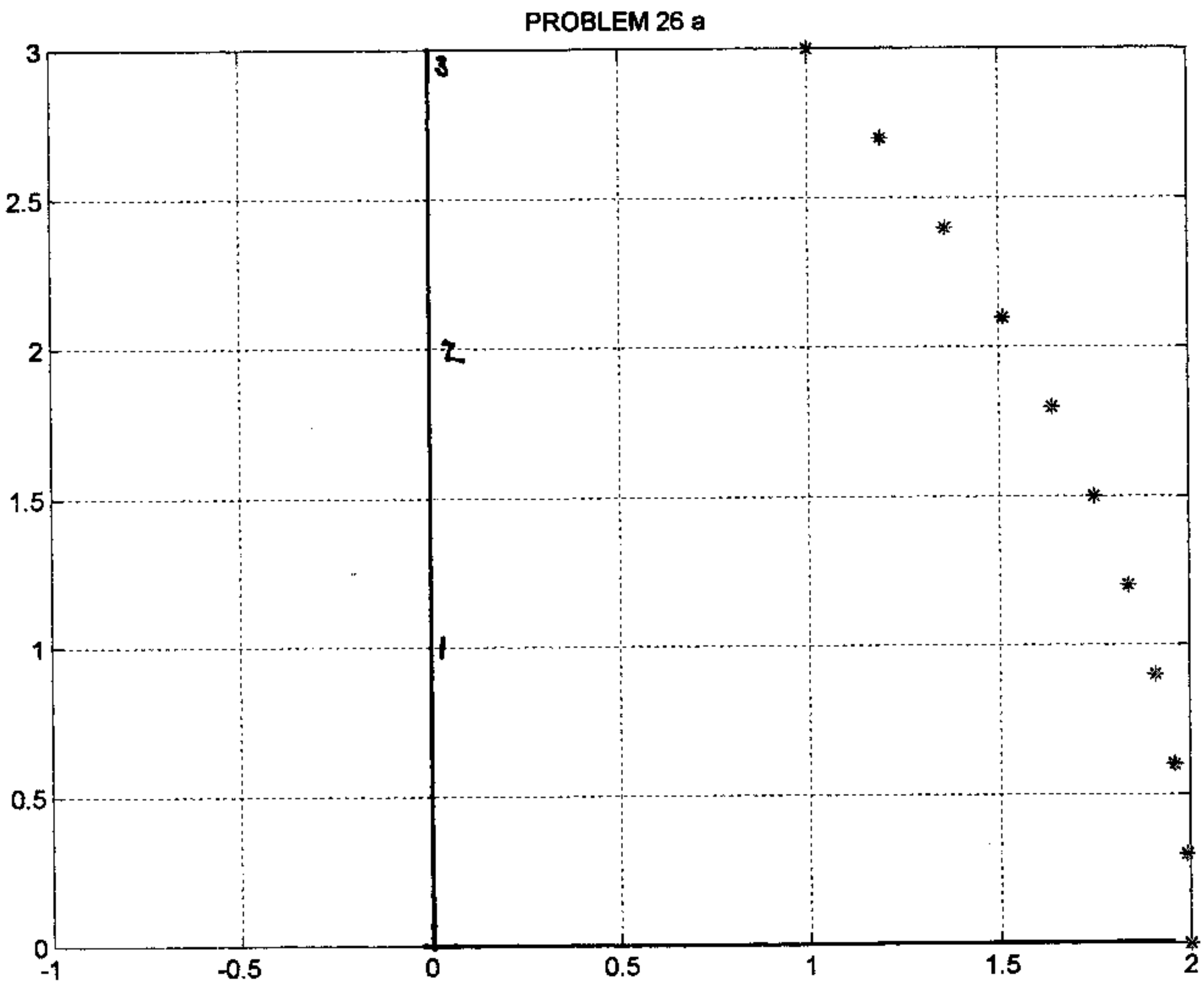
$\therefore f\left(\frac{1}{f(z)}\right) = \frac{1}{(z+i)+i} = \frac{1}{z+2i}$

25) $f(z+i) = \frac{1}{z+2i} = \frac{1}{x+i(y+2)} = \frac{x-i(y+2)}{x^2+(y+2)^2}$

$\therefore u = \frac{x}{x^2+(y+2)^2}, \quad v = -\frac{(y+2)}{x^2+(y+2)^2}$

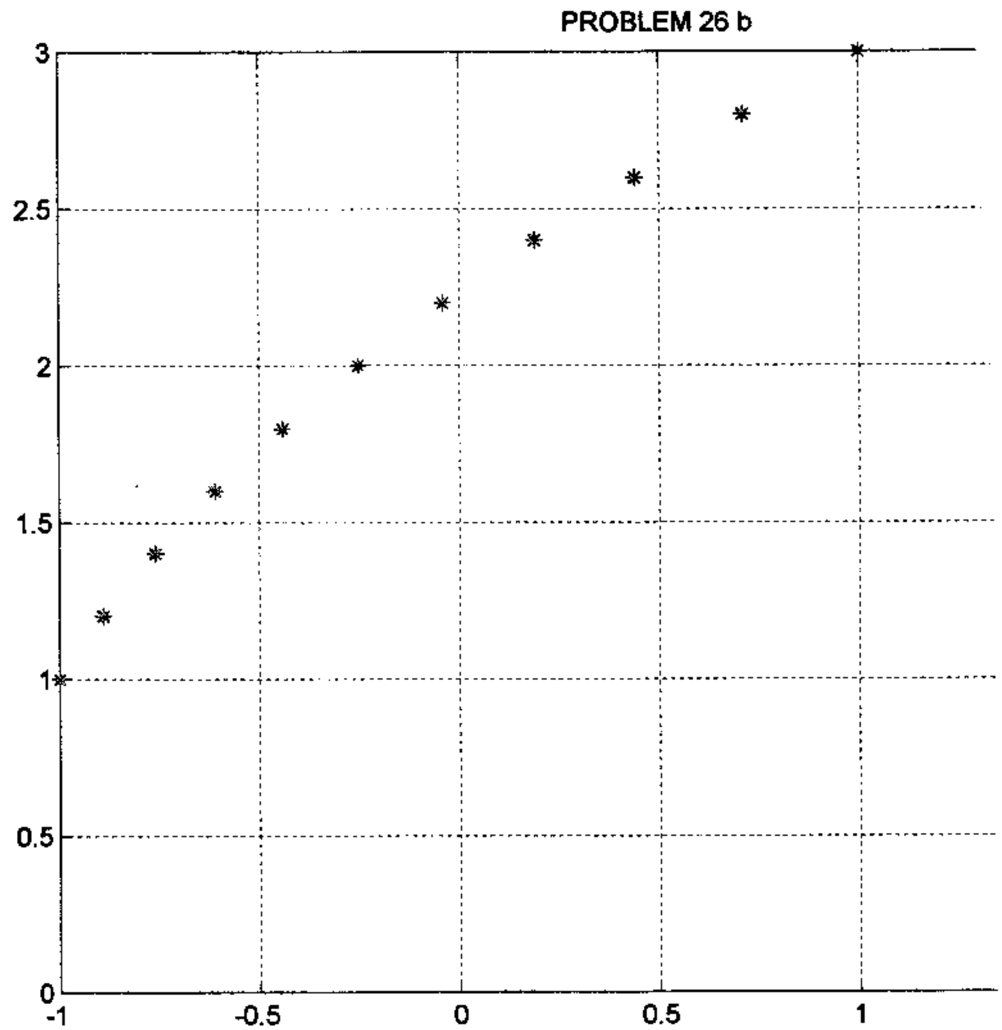
```
%prob26(a) SECTION 2.1
t=[0:.1:1]*i
z=1+t;

w=z.^2+z;
u=real(w);v=imag(w);
plot(u,v,'*');grid;axis([ -1 2 0 3])
title('PROBLEM 26 a')
```



```
%prob26(b) SECTION 2.1
t=[0:.1:1]
z=i+t;

w=z.^2+z;
u=real(w);v=imag(w);
plot(u,v,'*');axis([-1 2 0 3]);grid;
title('PROBLEM 26 b')
```



27a


```
% prob 27a, section 2.1
x=[-1:.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./(Z-3*i/2);
wm=abs(w);
meshz(X,Y,wm);hold on
title('Magnitude of 1/(z-3*i/2)')
```

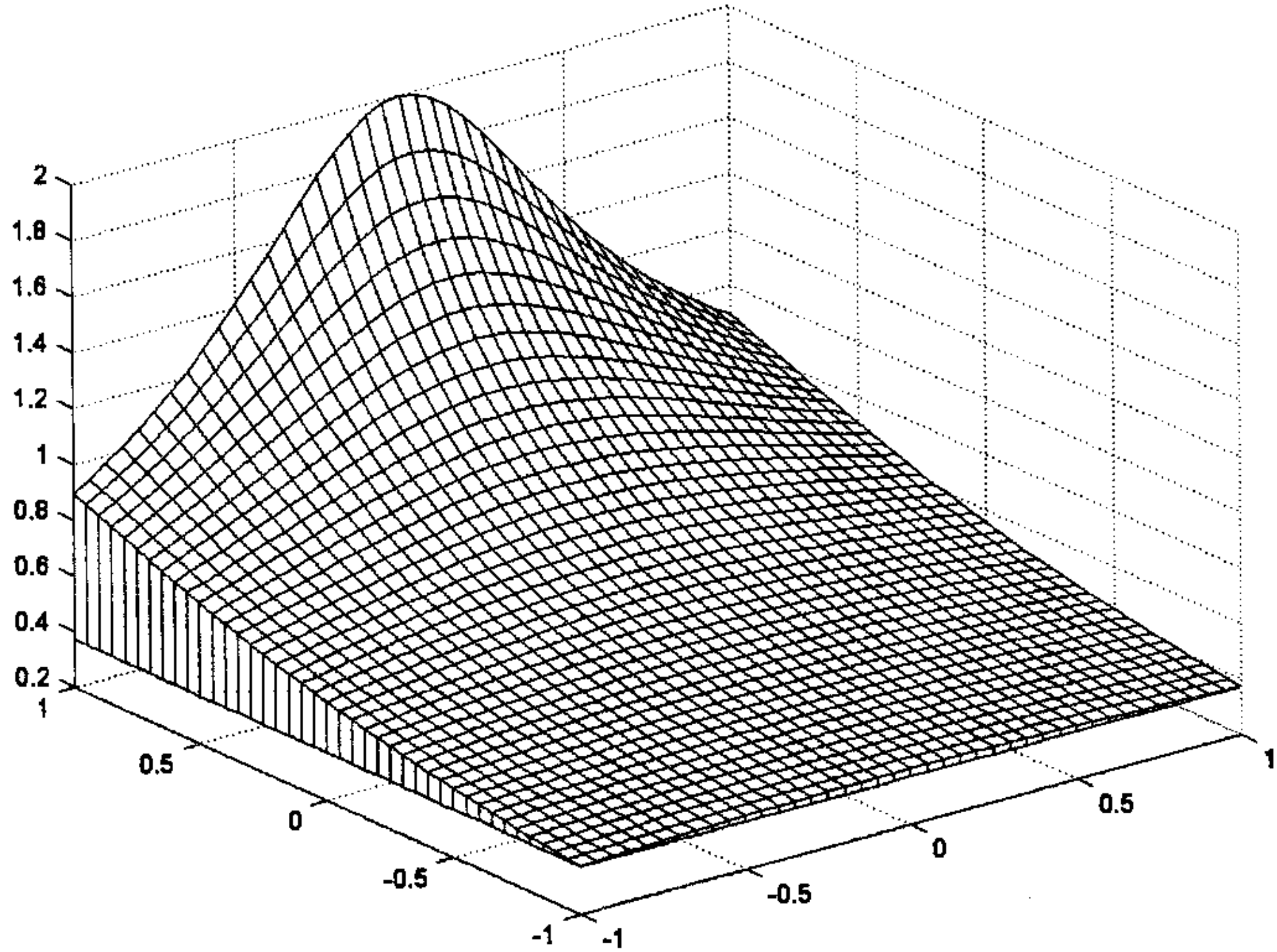
```
% prob 27(b), section 2.1
x=[-1:.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./(Z-3*i/2);
wm=real(w);
meshz(X,Y,wm);hold on
surf(X,Y,wm);
title('Re 1/(z-3*i/2)')
```

```
% prob 27(c), section 2.1
x=[-1:.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./((Z-3*i/2));
wm=imag(w);
meshz(X,Y,wm);hold on

title('Imag1/(z-3*i/2)')
```

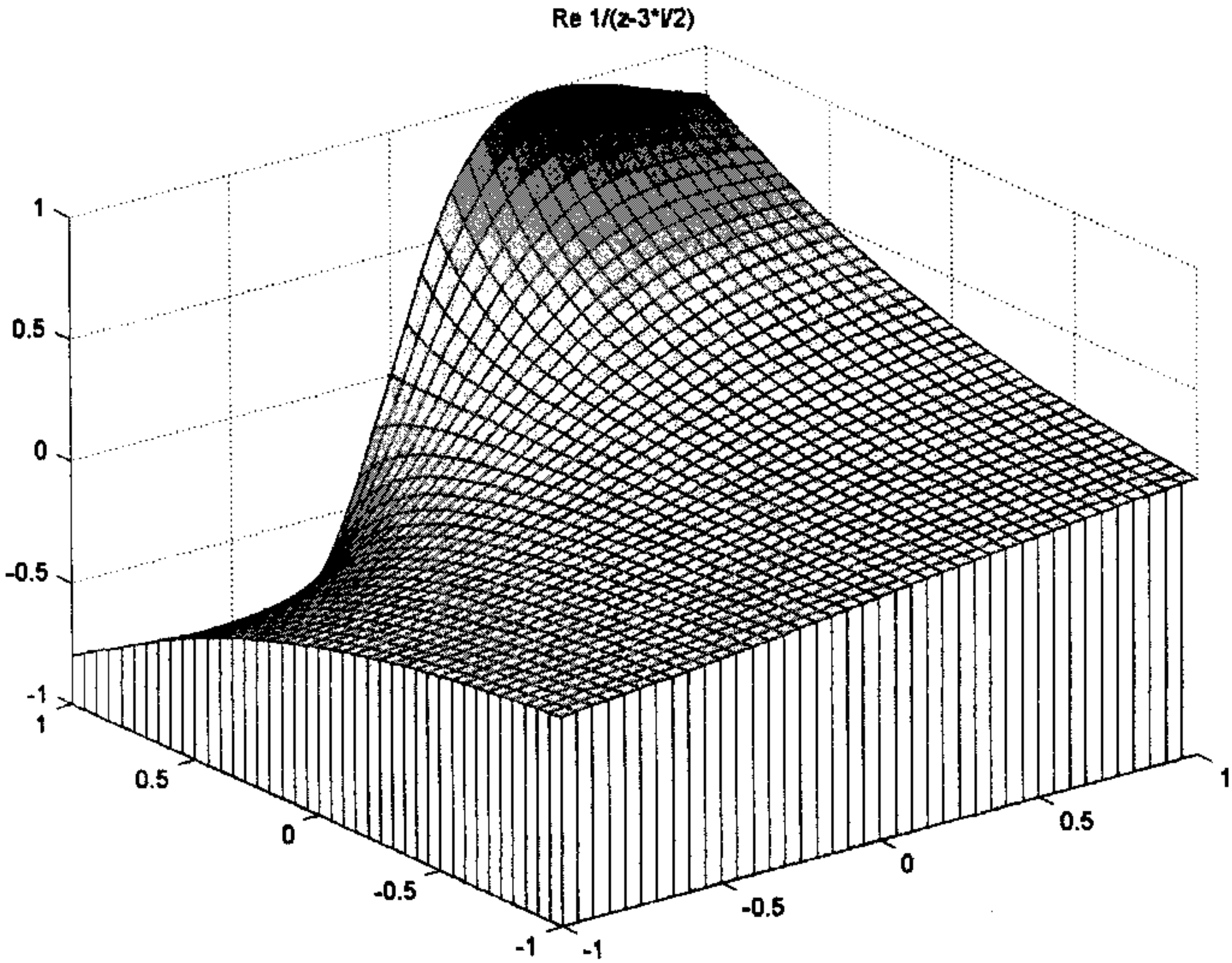
Problem 27 (a)

Magnitude of 1/(z-3*i/2)

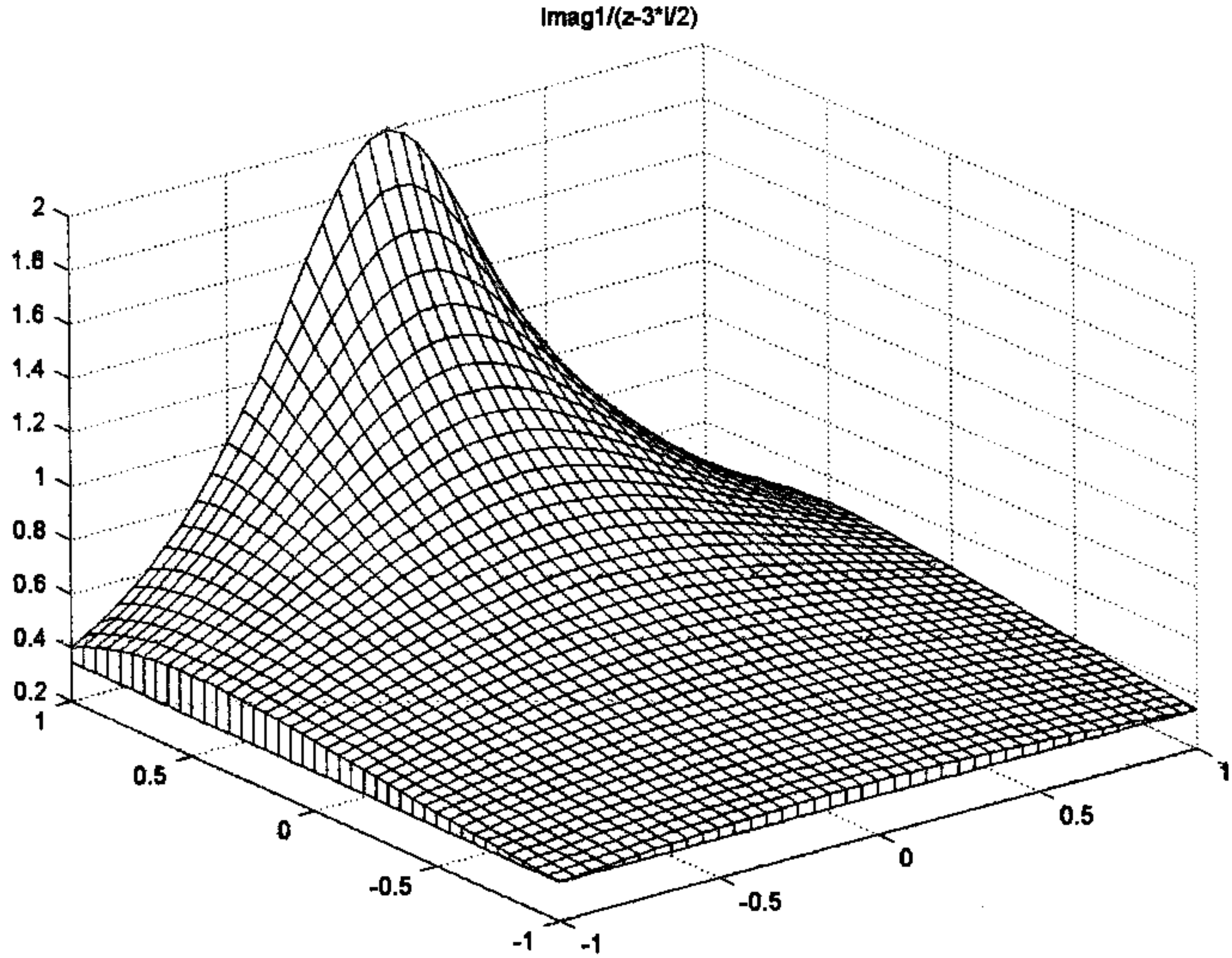


sec 2.1

)lem 27 (b)



problem 27 (c)



Sec 2.2

1) a) We need $|f(z) - f_0| < \epsilon$ for $0<|z-z_0| < \delta$

$f(z) = z$, $f_0 = z_0$

Thus need $|z-z_0| < \epsilon$ for $|z-z_0| < \delta$,
Just take $\delta = \epsilon$ (for example) and the requirement $|z-z_0| < \epsilon$ is satisfied in the deleted neighborhood of z_0 of radius δ.

b) $f(z)$ is defined at z_0
(and $f(z_0) = z_0$). Also $\lim_{z \to z_0} f(z) = z_0$
Finally, $\lim_{z \to z_0} f(z) = f(z_0)$ as required

2) Must first show $\lim_{z \to z_0} f(z) = c$. To do this we require $|f(z) - c| < \epsilon$ for $0 < |z-z_0| < \delta$
But $f(z) = c$ (all z). Thus $|f(z) - c| = 0$
and the requirement $|f(z) - c| < \epsilon$ is satisfied for all z. Now $f(z_0) = c$. Thus we have
$\lim_{z \to z_0} f(z) = f(z_0)$ and continuity is proved for all z.

3) $iz^3 = izzz$ is a product of continuous functions and is thus continuous.
$iz^3 + i$ is a sum of continuous funcs. and is therefore continuous

4) $1+i$ is constant, $\therefore$ continuous.
z^2 is continuous [product of 2 continuous funcs]
z^2+9 is sum of continuous funcs
$z^2+9=0$, $z = \pm 3i$

$\frac{1+i}{z^2+9}$, quotient of continuous functions is continuous except @ $\pm 3i$

5] z^4 being the product of continuous functions is continuous everywhere, z^2+3z+2, sum of continuous functions is continuous everywhere. $\frac{1+i}{z^2+3z+2}$ a quotient of continuous functions is continuous except where denom $=0$
$z^2+3z+2=0$, $[z=-1,\ z=-2]$. $z^4+\frac{(1+i)}{z^2+3z+2}$ is a sum of continuous functions [except where $z=-1$, $z=-2$]

6] Since $z+i$ is continuous for all z, so is $|z+i|$. $(1+i)z$ is continuous everywhere. $\therefore$ $|z+i|+(1+i)z$ [a sum of continuous functions] is continuous everywhere

7] Note that $x^2-y^2=Re(z^2)$ is the real part of a continuous function and is therefore continuous everywhere. $z^2+(x^2-y^2)$ is the sum of functions that are continuous everywhere

8] $z=x+iy$ is continuous everywhere
$x=Re(z)$, $y=Im(z)$ are continuous everywhere
$x-iy$ is the sum of continuous functions. $\therefore$ $\bar{z}$ is continuous everywhere. $\frac{z-i}{\bar{z}-i}$ is quotient of continuous functions and is therefore continuous except where $\bar{z}-i=0$, $\bar{z}=i$ $z=-i$

9] If $y=0$, $x>0$, $f(z)=\frac{\sin x}{x}\longrightarrow 1$ as $x\to 0$
If $x=0$, $y>0$ $f(z)=\frac{i\sin y}{-iy}=\frac{\sin y}{-y}\longrightarrow -1$ as $y\to 0$
If $x=y$, $f(z)=\frac{(1+i)\sin x}{(1-i)x}=\frac{i}{x}\sin x\longrightarrow i$ as $x\to 0$, $y\to 0$

[Thus limit does not exist.]

10) $f(z) = \dfrac{z-i}{z^2-3i-2} = \dfrac{(z-i)}{(z-2i)(z-i)}$

If $z \neq i$ we can write the preceding as $f(z) = \dfrac{1}{(z-2i)}$ $z \neq i$

Note that lim $z \to i$ $f(z) = \lim_{z \to i} \dfrac{1}{z-2i} = \dfrac{1}{i-2i} = i$

Since $\lim_{z \to i} f(z) = i$ and $f(i) = i$, and they agree, then $f(z)$ is continuous @ $z = i$

11) a) $f(z) = \dfrac{z^2-5z+6}{z^2-4} = \dfrac{(z-2)(z-3)}{(z-2)(z+2)}$

If we assume $z \neq 2$, we can rewrite the preceding as $f(z) = \dfrac{z-3}{z+2}$. The limit of this expression as $z \to 2$ is $-1/4$

∴ define $f(2) = -1/4$ to make $f(z)$ continuous at $z = 2$

b) $f(z) = \dfrac{z^4+10z^2+9}{z^2-4iz-3} = \dfrac{(z^2+9)(z^2+1)}{(z-3i)(z-i)}$

$= \dfrac{(z-3i)(z+3i)\ (z-i)(z+i)}{(z-3i)(z-i)}$. Now if $z \neq i$, or $z \neq 3i$

the preceding simplifies to $(z+3i)(z+i) = f(z)$

Thus $\lim_{z \to 3i} (z+3i)(z+i) = -24$

$\lim_{z \to i} (z+3i)(z+i) = -8$

∴ Take $f(3i) = -24$, $f(i) = -8$

12) a) Using definition: require

$$\left|\frac{z}{1+z} - 1\right| < \epsilon \qquad \text{provided } |z| > r(\epsilon)$$

or $\left|\frac{z - (1+z)}{1+z}\right| < \epsilon$ or $\left|\frac{1}{1+z}\right| < \epsilon$, $|z| > r$.

b) Recall that: $|1+z| \geq |z| - 1 > 0$ if $|z|>1$

Now take $|z| > r$, $r > 1$

$|1+z| \geq |z| - 1 > r - 1 > 0$ (triangle ineq.)

or $\frac{1}{|1+z|} < \frac{1}{r-1}$ where $r < |z|$, $|z| > 1$, $r > 1$

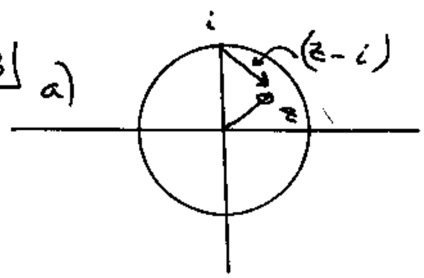
We require $\frac{1}{r-1} < \epsilon$ or $r - 1 > 1/\epsilon$

or $r > 1 + 1/\epsilon$

If $r > 1 + 1/\epsilon$, then $\frac{1}{r-1} < \epsilon$ and

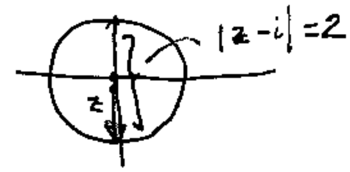
$\frac{1}{|1+z|} < \epsilon$ as required [if $|z| > r$]

13) a)



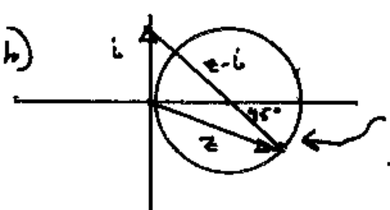
$|z-i|$ is max if

$z = -i$



max $|z-i| = 2$, This occurs if $z = -i$, $M = 2$

b)

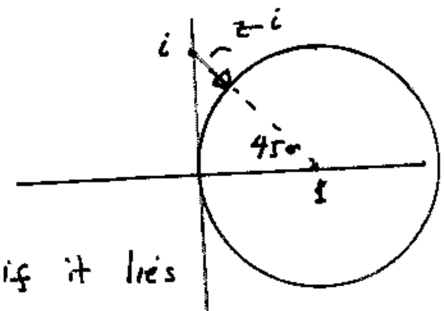


$|z-i|$ is max when $z-i$ goes thru center of circle

max is where $x = 1 + 1/\sqrt{2}$, $y = -1/\sqrt{2}$

$$|z-i| = \sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}\left[1+\frac{1}{\sqrt{2}}\right] = 1+\sqrt{2} = M$$

13 (c)



$|z-i|$ is min if it lies along line going thru center of circle

Occurs if $x = 1 - \frac{1}{\sqrt{2}}$, $y = 1/\sqrt{2}$

what is $|z-i| = \sqrt{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{2}\left[1-\frac{1}{\sqrt{2}}\right] = \sqrt{2}-1 \quad \therefore \; M = \frac{1}{|z-i|}\Big|_{\substack{z=i \\ min}} = \frac{1}{\sqrt{2}-1}$

14 (a) $z^2 = x^2 - y^2 + i\,2xy$

$Im(z^2) = 2xy$ is continuous

$\therefore$ xy is continuous.

(b) The real part of g is xy which we know to be continuous

x is continuous $= Re(z)$ $\quad$ y is continuous $= Im(z)$

$\therefore$ $x+y$ is continuous, and the imaginary part of $g(x,y)$ is continuous. Since the real part is continuous too, $g(x,y)$ must be continuous.

15 Consider $g(z) = 1 + \frac{1}{z^2}$ which does not have a limit as $z \to 0$ [becomes unbounded] and $h(z) = 1 - 1/z^2$ which does not have a limit as $z \to 0$, for the same reason.
Let $f(z) = g(z) + h(z) = 2$ has a limit everywhere.

16) $g(z) = 1$ if $x \ge 0$; $g(z) = -1$ if $x < 0$ } no limit if $x=0$

$h(z) = -1$ if $x \ge 0$; $h(z) = 1$ if $x < 0$ } has no limit if $x=0$

$g(z)\,h(z) = -1$ if $x \neq 0$, but has a limit of -1 everywhere in complex plane.

17) Assume $f(z)$ has a limit as $z \to z_0$

$f(z) = g(z) + h(z)$, $f(z) - g(z)$

has a limit as $z \to z_0$ (see Eq. 2.2-10a)

But $f(z) - g(z) = h(z)$ and $h(z)$ by assumption does not have a limit as $z \to z_0$.

Thus we contradict ourselves by assuming that $g(z) + h(z)$ has a limit as $z \to z_0$

18) a) want $\left|\frac{1}{z}\right| > \rho$ for all $0 < |z| < \delta$

If $\left|\frac{1}{z}\right| > \rho$, $|z| < 1/\rho$. ∴ If take $\delta = 1/\rho$, we have $\left|\frac{1}{z}\right| > \rho$ if $0 < |z| < \delta = 1/\rho$

b) need $\left|\frac{1}{z-i}\right|^2 > \rho$ for $0 < |z-i| < \delta$

If $\left|\frac{1}{z-i}\right|^2 > \rho$, then $|z-i|^2 < 1/\rho$, $|z-i| < \sqrt{1/\rho}$

∴ take $\delta = \sqrt{1/\rho}$

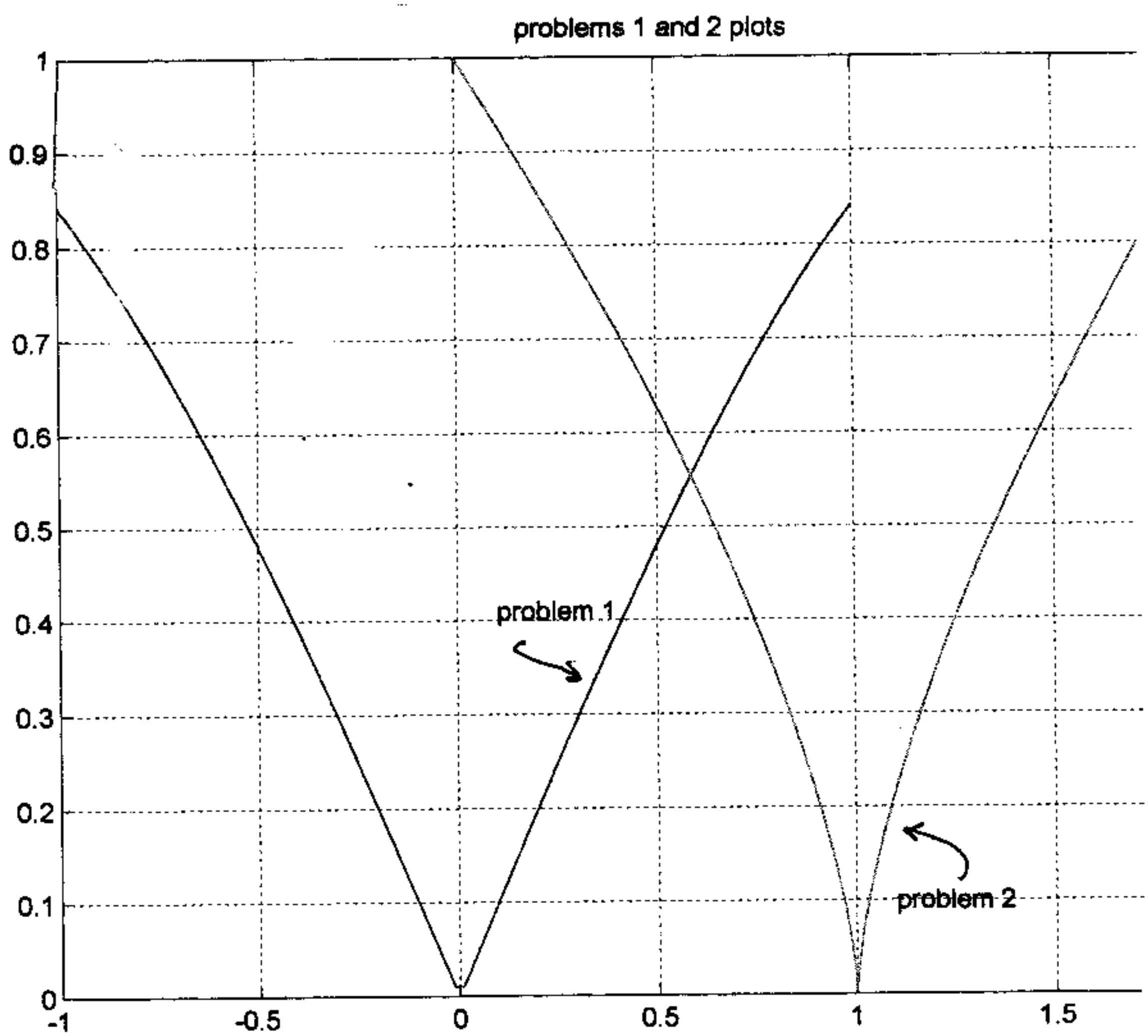
c) In real calculus, one distinguishes between ∞ and $-\infty$. Thus $\lim_{x \to 0+} 1/x = \infty$ and $\lim_{x \to 0-} 1/x = -\infty$ but $\lim_{x \to 0} \frac{1}{x}$ does not exist [compare right and left hand limits] See Thomas' Calculus, 10th ed, p. 115.

d) Need $|z^2| > \rho$ for $|z| > r$. $|z|^2 > \rho$ implies $|z| > \sqrt{\rho}$

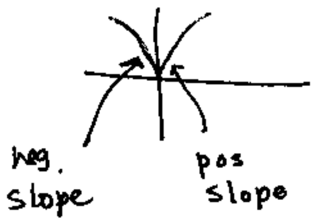
∴ $r = \sqrt{\rho}$

sec 2.3

```
%code for plots in probs 1 and 2
x1=linspace(-1,1,100);
y1=sin(abs(x1));
x2=linspace(0,2,1000);
y2=(abs(x2-1)).^(2/3);
% note, you cannot use (x2-1).^2/3 as it will not give
%the real root, but a complex one if x<1.The plot
% would be of real part
plot(x1,y1,x2,y2);grid;text(.1,.41,'problem 1')
text(1.1,.1,'problem 2')
title('problems 1 and 2 plots')
```



Sec 2.3

1] 

$\sin|x|$ is continuous at $x=0$ but has no deriv. at $x=0$ [See previous pg.]

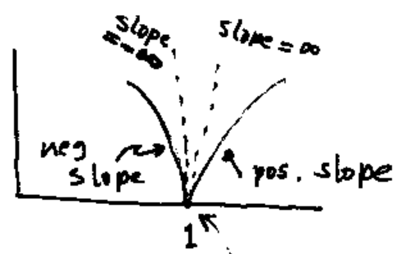
2] $(x-1)^{2/3} = \left[\sqrt[3]{x-1}\right]^2$ if $x \geq 1$ [real rt]

$(x-1)^{2/3} = \left[\sqrt[3]{|x-1|}\right]^2 \angle \frac{2\pi}{3} + \frac{2k\pi}{3}$ $\quad k=0,1,2$

if $x \leqslant 1$

put $k=2$ for real root

$= \left[\sqrt[3]{|x-1|}\right]^2$



continuous at $x=1$, but $f'(1)$ does not exist

no deriv at $x=1$

3]

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{|\Delta z| \angle -\arg\Delta z}{|\Delta z| \angle \arg\Delta z} = \lim_{\Delta z \to 0} 1 \angle -2\arg(\Delta z)$$

The preceding result depends on $\arg(\Delta z)$ and should be independent of direction ($\arg \Delta z$) if the limit is to exist.

sec 2.3

4] $f(z) = c$ $\qquad \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = 0$

for all z. Deriv. exists for all z.

Can use C-R eqns too.

5] $u = 1, \quad v = y$

$\dfrac{\partial u}{\partial x} = 0, \quad \dfrac{\partial v}{\partial y} = 1 \qquad 0 = 1$ nowhere

Deriv. exists nowhere.

6] $\dfrac{d}{dz} z^6 = 6z^5$ for all z [See Eq. (2.3-4)]

7] $\dfrac{d}{dz} z^{-5} = -5z^{-6} \qquad$ all $z \neq 0$

8] $u = y, \; v = x \qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 0$

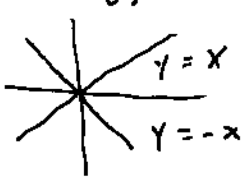
$\dfrac{\partial u}{\partial y} = 1, \quad \dfrac{\partial v}{\partial x} = 1 \qquad \dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

No deriv. anywhere

9] $u = xy, \; v = xy$

$\dfrac{\partial u}{\partial x} = y, \quad \dfrac{\partial v}{\partial y} = x \qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow x = y$

$\dfrac{\partial u}{\partial y} = x, \quad \dfrac{\partial v}{\partial x} = y \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow x = -y$



only solution $x = 0, y = 0$, or $z = 0$

10] $u = x^2, \quad v = y, \qquad \dfrac{\partial u}{\partial x} = 2x, \quad \dfrac{\partial v}{\partial y} = 1$

$2x = 1, \; x = 1/2 \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = 0$

$\therefore$ Have deriv if $x = 1/2, \; -\infty < y < \infty$

or $\operatorname{Re}(z) = 1/2$

11| $f(z) = x + i|y|$ Suppose $y > 0$

$f(z) = x + iy = z$ has a derivative

Thus $f(z)$ has a derivative for $\operatorname{Im}(z) > 0$

Suppose $y < 0$ $f(z) = x - iy$

$u = x,\ v = -y$ $\quad \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$, No deriv. if $y < 0$

Suppose $y = 0$, is there a deriv?

Let $z_0 = x_0$, Take $\Delta z = i\Delta y$

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta y \to 0} \frac{x_0 + i|\Delta y| - x_0}{i\Delta y} = \lim_{\Delta y \to 0} \frac{|\Delta y|}{\Delta y}$$

$= 1$ if $\Delta y > 0$, and $= -1$ if $\Delta y < 0$. Thus the limit does not exist if $y = 0$. To summarize:

$x + i|y|$ has a deriv. if and only if $\operatorname{Im}(z) > 0$ or $y > 0$

12| $u = e^x,\ v = e^{2y}$, $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Means $e^x = 2e^{2y}$, $\quad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$ is satisfied.

$e^x = 2e^{2y}$ $\quad e^{x-2y} = 2$, $\quad x - 2y = \operatorname{Log} 2$ [natural log]

$y = \frac{x}{2} - \frac{1}{2}\operatorname{Log} 2$ deriv exists on this line.

13| $u = y - 2xy$, $\quad v = -x + x^2 - y^2$

$\frac{\partial u}{\partial x} = -2y$, $\quad \frac{\partial v}{\partial y} = -2y$ $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ all x, y

$\frac{\partial u}{\partial y} = 1 - 2x$, $\quad \frac{\partial v}{\partial x} = -1 + 2x$, $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

for all x, y, Thus deriv. exists everywhere in z plane.

14] $f(z) = (x-1)^2 + iy^2 + z^2$. Note: z^2 has a deriv for all z, ∴ we must find where $(x-1)^2 + iy^2$ has a deriv.

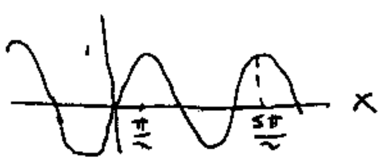
$u = (x-1)^2$, $v = y^2$, $\frac{\partial u}{\partial x} = 2(x-1)$

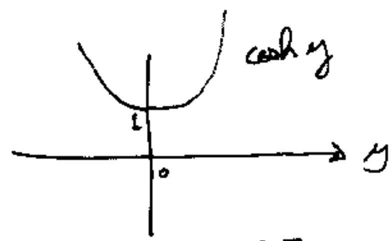
$\frac{\partial v}{\partial y} = 2y$ $\quad 2(x-1) = 2y \quad (y = x-1)$, Note: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ all z. Thus the derivative of $(x-1)^2 + iy^2 + z^2$ exists only on the line $y = x-1$

15] $u = \cos x$, $v = -\sinh y$

$\frac{\partial u}{\partial x} = -\sin x$, $\frac{\partial v}{\partial y} = -\cosh y$ $\quad$ Note $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$

∴ $\sin x = \cosh y$





For solution, $\boxed{y=0,}$ $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

or $x = -\frac{3\pi}{2}$, $x = -\frac{7\pi}{2}$, $x = -\frac{11\pi}{2}$ equivalently: $\boxed{x = \frac{\pi}{2} \pm 2n\pi \quad n = 0, 1, 2, \dots}$

16] Suppose $f(z) = 1/z$, $|z| > 1$, $f'(z) = -1/z^2$ deriv. exists. Suppose $f(z) = z$, $|z| < 1$, $f'(z) = 1$. Let $1/z = z$, $z = \pm 1$.

On the circle $|z| = 1$, $f(z)$ is discontinuous except at $z = \pm 1$, so there is no derivative on the circle $|z| = 1$, except possibly at $z = \pm 1$. Now suppose you compute $f'(z)$ at $z = 1$, Take $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{f(x+\Delta x) - f(x)}{\Delta x}\Big|_{z=1}$ get 1 if $\Delta x < 0$, get -1 if $\Delta x > 0$

so the limit does not exist. A similar argument holds at $z = -1$. i.e. the limit does not exist.

$f(z)$ has a derivative for all z except on circle $|z| = 1$.

17] $\bar{z}$ has a derivative nowhere

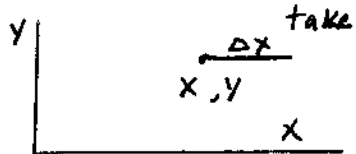
let $\boxed{g(z) = z + \bar{z}} = 2x$, let $\boxed{h(z) = z - \bar{z}} = 2iy$

$g(z)$ and $h(z)$ have derivatives nowhere.

$f(z) = g(z) + h(z) = 2z$ has a derivative, all z.

18] If $\dfrac{d^2 f}{dz^2}$ exists, then $\dfrac{df}{dz}$ must exist. We have from Eq. (2.3-6) that $\dfrac{df}{dz} = \dfrac{\partial u}{\partial x}(x,y) + i\dfrac{\partial v}{\partial x}(x,y)$

take $\Delta z = \Delta x$



Thus $\dfrac{d^2 f}{dz^2} =$

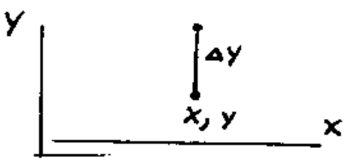
Thus $\dfrac{d^2 f}{dz^2} = \lim\limits_{\Delta x \to 0}\left[\dfrac{\dfrac{\partial u}{\partial x}(x+\Delta x, y) + i\dfrac{\partial v}{\partial x}[x+\Delta x, y] - \dfrac{\partial u}{\partial x}(x,y) - i\dfrac{\partial v}{\partial x}(x,y)}{\Delta x}\right]$

$$\frac{d^2 f}{dz^2} = \lim_{\Delta x \to 0}\left[\frac{\dfrac{\partial u}{\partial x}(x+\Delta x, y) - \dfrac{\partial u}{\partial x}(x,y)}{\Delta x} + i\,\frac{\dfrac{\partial v}{\partial x}(x+\Delta x, y) - \dfrac{\partial v}{\partial x}(x,y)}{\Delta x}\right]$$

$= \dfrac{\partial^2 u}{\partial x^2} + i\dfrac{\partial^2 v}{\partial x^2}$. (first required identity)

Have from (2.3-8) that $\dfrac{df}{dz} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$

Take $\Delta z = i\Delta y$



Thus $\dfrac{d^2 f}{dz^2} = \lim\limits_{\Delta y \to 0} \dfrac{\dfrac{\partial}{\partial y}v(x, y+\Delta y) - i\dfrac{\partial}{\partial y}u(x, y+\Delta y) - \dfrac{\partial}{\partial y}v(x,y) + i\dfrac{\partial}{\partial y}u(x,y)}{i\Delta y}$

$= \lim\limits_{\Delta y \to 0} -\left[\dfrac{\dfrac{\partial}{\partial y}u(x, y+\Delta y) - \dfrac{\partial}{\partial y}u(x,y)}{\Delta y}\right] - i\left[\dfrac{\dfrac{\partial}{\partial y}v(x, y+\Delta y) - \dfrac{\partial}{\partial y}v(x,y)}{\Delta y}\right]$

$= -\dfrac{\partial^2 u}{\partial y^2} - i\dfrac{\partial^2 v}{\partial y^2}. = \dfrac{d^2 f}{dz^2}$ (second required identity).

19

We must show that $\lim_{z \to z_0} f(z) = f(z_0)$

or, equivalently, that $\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$

Now $\lim_{\Delta z \to 0} \left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \right] \lim_{\Delta z \to 0} \Delta z =$

$f'(z_0) \lim_{\Delta z \to 0} \Delta z = 0$. Now apply Theorem 1(b)

$\lim_{\Delta z \to 0} \left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \right] \lim_{\Delta z \to 0} \Delta z = \lim_{\Delta z \to 0} \left[\frac{[f(z_0+\Delta z) - f(z_0)] \Delta z}{\Delta z} \right]$

$= \lim_{\Delta z \to 0} \left[f(z_0+\Delta z) - f(z_0) \right]$. We have just proved this to be zero.

Thus $\lim_{\Delta z \to 0} f(z_0 + \Delta z) - f(z_0) = 0$ or

$\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$ which is equivalent

to $\lim_{z \to z_0} f(z) = f(z_0)$, as required.

Section 2.4

1) $u = xy, \quad v = xy + x$

$\frac{\partial u}{\partial x} = y \quad \frac{\partial v}{\partial y} = x$

$\therefore y = x$

$\frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = y + 1$

$\therefore \quad x = -y - 1 \qquad x = -x - 1 \qquad x = -1/2$

$y = -1/2$

deriv at $\boxed{z = \frac{-1}{2} - i/2}$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad y + i\ (y+1)\Big|_{y=-1/2} = \frac{-1}{2} + \frac{i}{2} = f'(z)$

f(z) not analytic anywhere, since its deriv. does not exist in a domain – only at one point.

2) a) Note $1/z$ has a deriv for all $z \neq 0$

Look @ deriv. of $(x-1)^2 + i\,xy$, $u = (x-1)^2, v = xy$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad 2(x-1) = x \qquad 2x - 2 = x$

$x = 2$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad 0 = -y, \quad y = 0$

Deriv only at $z = 2$

$f'(z) = -1/z^2 + \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ where $u = (x-1)^2$

$v = xy$

$f'(z) = -\frac{1}{z^2} + 2(x-1) + iy$

$f'(2) = \frac{-1}{4} + 2 + i0 = 1\,3/4 = 7/4$

b) Function nowhere analytic, since deriv. does not exist in a domain.

3) a) $z^3 + z^2 + 1$ is a sum of entire functions and is $\boxed{\text{analytic everywhere}}$

b) $f'(z) = 3z^2 + 2z$

$f'(1+i) = 3[1+i]^2 + 2 + 2i = 2 + 8i$

sec 24, contid

4] z^2 has a deriv. everywhere.

a) What about $(x-1)^2 + i\,(y-1)^2$

$$u = (x-1)^2, \quad v = (y-1)^2$$

$$\frac{\partial u}{\partial x} = 2(x-1) = \frac{\partial v}{\partial y} = 2(y-1)$$

$$(x-1) = (y-1)$$

$$x = y$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad \text{is satisf. everywhere}$$

$z^2 + (x-1)^2 + i\,(y-1)^2$ has a deriv. only along the line $x = y$.

b)

deriv. along here.
Not a domain

function is nowhere analytic

For any z_0 on this line, any neighborhood of z_0 will contain pts where $f'(z_0)$ does not exist.

c) $$f'(z) = 2z + \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 2z + 2(x-1) = 2 + 2i \quad \text{at } 1,1$$

5 (a) $$f(z) = \frac{1}{e^{2x}\cos 2y + i\,e^{2x}\sin 2y}$$

Note: the denom $\neq 0$ proof:

$$e^{2x}\cos 2y + i\,e^{2x}\sin 2y = 0$$

$$\cos 2y + i\sin 2y = 0 \qquad \text{since } e^{2x} \neq 0$$

$$|\cos 2y + i\sin 2y| = 1 \qquad \neq 0$$

$$f(z) = \frac{e^{2x}\cos 2y - i\,e^{2x}\sin 2y}{[e^{2x}\cos 2y]^2 + [e^{2x}\sin 2y]^2} =$$

$$= \frac{e^{2x}\cos 2y - i\,e^{2x}\sin 2y}{e^{4x}[\cos^2 + \sin^2]} = e^{-2x}\cos 2y - i\,e^{-2x}\sin 2y$$

5(a) cont'd

$$u = e^{-2x}\cos 2y, \quad v = -e^{-2x}\sin 2y$$

$$\frac{\partial u}{\partial x} = -2e^{-2x}\cos 2y = \frac{\partial v}{\partial y} = -2e^{-2x}\cos 2y$$

$$\frac{\partial u}{\partial y} = -2e^{-2x}\sin 2y = -\frac{\partial v}{\partial x}$$

These are satisfied everywhere. Have an entire function.

5(b) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -2e^{-2x}\cos 2y + i2e^{-2x}\sin 2y$

at $1+i\pi/4$ this equals: $-2e^{-2}\cos\frac{\pi}{2} + i\,2e^{-2}\sin\frac{\pi}{2}$

$= i2e^{-2}$

6] a) $f(z) = z\,[\cos x\cosh y - i\sin x\sinh y]$

Since z is an entire function, we need only show that $\cos x\cosh y - i\sin x\sinh y$ is entire. Product of entire functions is entire. Take $u = \cos x\cosh y$, $v = -\sin x\sinh y$

$\frac{\partial u}{\partial x} = -\sin x\cosh y = \frac{\partial v}{\partial y}$ satisf. everywhere

$\frac{\partial v}{\partial x} = -\cos x\sinh y = -\frac{\partial u}{\partial y}$ satisf. everywhere

$\therefore$ expression in brackets is entire function

b) $\frac{df}{dz} = [\cos x\cosh y - i\sin x\sinh y] +$

$z\frac{d}{dz}[\cos x\cosh y - i\sin x\sinh y]$

$= [\cos x\cosh y - i\sin x\sinh y] + z\,[\underbrace{-\sin x\cosh y - i\cos x\sinh y}_{\text{used } \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}]$

6] cont'd

For $f'(i)$ put $x=0,\ y=1,\ z=i$

$$f'(i) = [\cosh 1] + i[-i \sinh 1]$$

$$= \cosh 1 + \sinh 1 = e$$

7] Let $u = \sin x \cosh y,\ v = \cos x \sinh y$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\sin x \sinh y = -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{satisfied all } x, y$$

$\therefore\ \sin x \cosh y + i \cos x \sinh y$ is an entire function

$$\frac{d}{dz}[\sin x \cosh y + i \cos x \sinh y]^5$$

$= 5[\sin x \cosh y + i \cos x \sinh y]^4$ times

$$\frac{d}{dx}[\sin x \cosh y + i \cos x \sinh y]$$

$$= 5[\sin x \cosh y + i \cos x \sinh y]^4 \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

$$= 5[\sin x \cosh y + i \cos x \sinh y]^4 [\cos x \cosh y - i \sin x \sinh y]$$

put $x = \pi,\ y = 2$

$$5[\sin\pi \cosh 2 + i \cos\pi \sinh 2]^4 [\cos\pi \cosh 2 - i \sin\pi \sinh 2]$$

$$= 5[\sinh 2]^4(-\cosh 2)$$

Sec 2.4 cont'd

8 (a) z is entire

$(1+iz)$ is entire func.

$\dfrac{z}{(1+iz)^4}$ is analytic except where

$1+iz = 0$ or $z = i$

b) $f'(z) = \dfrac{(1+iz)^4 - (z)4(1+iz)^3 i}{(1+iz)^8}$

put $z = -i$

$f'(-i) = \dfrac{2^4 + (i)^2(4)(2)^3}{2^8} = \boxed{\dfrac{-1}{16}}$

9| L'Hopital's Rule applies since $g(i) = 0 = h(i)$

and $h'(i) = 2z - 3i \neq 0$ if $z = i$.

$\lim_{z \to i} \dfrac{g}{h} = \lim_{z \to i} \dfrac{1+2z}{2z-3i} = \dfrac{1+2i}{2i-3i} =$

$\dfrac{1+2i}{-i} = \boxed{i-2}$

10| L'Hopitals Rule applies since $g(i) = 0 = h(i)$

and $h'(i) = 3z^2+1\Big|_{z=i} = -2 \neq 0$

$\lim_{z \to i} \dfrac{g}{h} = \lim_{z \to i} \dfrac{3z^2}{3z^2+1}\Big|_{z=i} = \dfrac{-3}{-2} = \dfrac{3}{2}$

11] Let $f(z) = g(z) + h(z)$.
Assume $f(z)$ has a deriv. at z_0.
$f(z) - g(z)$ has a deriv at z_0
which equals $f'(z_0) - g'(z_0)$
[See Theorem 4, part (a)]
But $f'(z_0) - g'(z_0) = h'(z_0)$ which is known
not to exist. ∴ you have obtained a
contradiction by assuming that $f'(z_0)$ exists

12] Let $g(z) = x + i2y$,
$h(z) = -iy$
$g(z) + h(z) = x + iy$ which has
a derivative everywhere, is thus analytic.
Note that neither $g(z)$ or $h(z)$
satisfies the C-R eqns.
anywhere is complex plane

13] (a) $f(z) = g(z)\,h(z)$
$h(z) = f(z) / g(z)$ assuming $g(z) \neq 0$

Now according to theorem 5 the quotient
$f(z)/g(z)$ must be analytic at z_0. But $f(z)/g(z)$
$= h(z)$ which by assumption is not analytic
at z_0. ∴ you have obtained a
contradiction by assuming that
$g(z)\,h(z)$ is analytic.

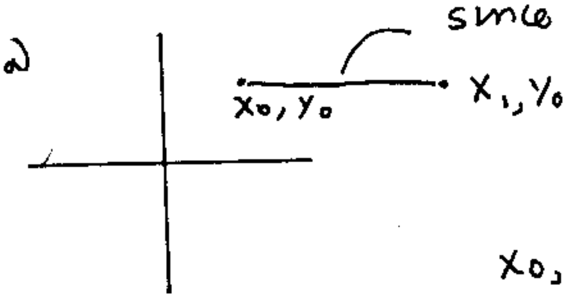
14] Try $g(z) = \frac{z}{x} = 1 + i\frac{y}{x}$

which is nowhere analytic.

Try $h(z) = x$ which is nowhere analytic.

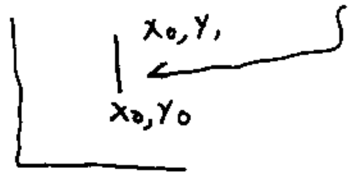
$g(z)h(z) = z$ which is entire.

15] a)

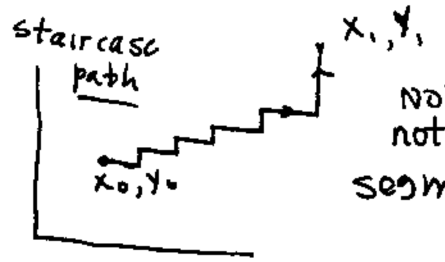


since $\frac{\partial \phi}{\partial x} = 0$ along here, ϕ must not change between x_0, y_0 and x_1, y_0

The same argument works here



since $\frac{\partial \phi}{\partial y} = 0$ ϕ does not change.



Note that ϕ does not change on any segment of the path.

b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ← but $v = 0$ because func. is real

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ u is constant from argument in part (a).

Note: if $f(z)$ is imag. $f(z) = 0 + iv$, $u = 0$ and an analogous argument applies: v is constant.

16] $f(z) = u + iv$ is analytic. Assume $g(z) = u - iv$ is too. $f+g$ must be analytic, $2u(x,y) = f+g$ is real
∴ u is constant. Now look at $f - g = 2iv$. $f-g$ the difference of 2 analytic functions is analytic and purely imaginary. ∴ $v(x,y)$ is constant. This completes the proof that f and g are constant

17] $|u+iv| = k$, $u^2+v^2 = k^2$

$$\frac{k^2}{(u+iv)(u-iv)} = 1, \quad \frac{k^2}{u+iv} = u - iv$$

Now $\frac{k^2}{u+iv}$ is analytic. Thus $u-iv$ is analytic, However, if both $u+iv$ and $u-iv$ are analytic, then u and v are constants (see prev. problem)
Thus $f(z)$ is constant.

18] (a) $f(z) = u(x,y) + i\,v(x,y)$
$f(\bar{z}) = \bar{f}(z) = u - iv$ by assumption
Now, assume $f(\bar{z})$ is analytic
$f(z) + f(\bar{z})$ is analytic
∴ $f(z) + \bar{f}(z)$ is analytic $= 2u(x,y)$ (a real func.)
By problem 15, $u(x,y)$ is constant.
$f(z) - f(\bar{z})$ is analytic $= f(z) - \bar{f}(z) = 2i\,v(x,y)$
But $v(x,y)$ must be constant, by prob 15 (b)
∴ $f(z)$ is constant, since u and v are.
(b) next pg.

18 (b) $f(z) = z^3 + z$ is entire and not constant. $f(\bar{z}) = (\bar{z})^3 + \bar{z} = \overline{(z^3+z)}$

Since $f(\bar{z}) = \bar{f}(z)$ and $f(z)$ is entire and not constant then $f(\bar{z}) = (\bar{z})^3 + \bar{z}$ is not analytic anywhere in complex plane

19

a) Assume $(\bar{z}+1)^2$ has a derivative.

Then, following the chain rule

$$\frac{d}{dz}(\bar{z}+1)^2 = \frac{d}{dz}(z+1)^2\Big|_{\bar{z}} \cdot \frac{d}{dz}\bar{z} = 2(\bar{z}+1)\frac{d\bar{z}}{dz}$$

But $\frac{d\bar{z}}{dz}$ does not exist. However if $\bar{z}+1=0$ or $z=-1$, the derivative might still exist. But the existence of the derivative at one point will not yield analyticity. (require a deriv. in a domain).

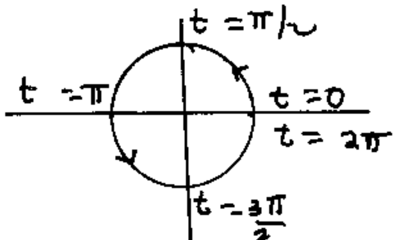
b) Assume $(\bar{z})^3$ has a derivative. Then, following the chain rule we have:

$$\frac{d}{dz}(\bar{z})^3 = \frac{d}{dz}z^3\Big|_{z=\bar{z}} \cdot \frac{d}{dz}\bar{z} = 3\bar{z}^2\frac{d}{dz}\bar{z}$$

But $d\bar{z}/dz$ does not exist. Thus the above derivative $\frac{d}{dz}(\bar{z})^3$ does not exist except possibly where $3\bar{z}^2 = 0$ $[z=0]$. But the existence of the derivative at one point will not produce analyticity.

201

a) $f(t) = \cos t + i \sin t = \text{cis}\, t = 1 \angle t$. As t goes from 0 to 2π we trace out a unit circle.

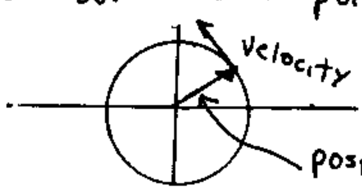


b) $f'(t) = -\sin t + i \cos t =$

$\cos\left(t + \frac{\pi}{2}\right) + i \sin\left(t + \frac{\pi}{2}\right) = 1 \angle t + \frac{\pi}{2}$

Note that the vectors $1 \angle t$ and $1 \angle t + \pi/2$ are perpendicular to each other.

c) The position and velocity vectors are at right angles because the motion is along a circle. The velocity vector is tangent to the circle at each point



(c) cont'd $f'(t) = -\sin t + i \cos t$

$f''(t) = -\cos t - i \sin t =$

$= \cos(t+\pi) + i \sin(t+\pi) = 1 \angle t+\pi$

The velocity is $1 \angle t+\pi/2$. The angles differ by $\pi/2$, so the vectors are at rt. angles.

(d)

```
% problem 20 sec 2.4
t=linspace(0,2*pi,100);
f=cos(t)./(1+.5*(cos(t)+i*sin(t)));
x=real(f);
y=imag(f);
plot(x,y);hold on
t1=linspace(0,2*pi,9);
f1=cos(t1)./(1+.5*(cos(t1)+i*sin(t1)));
x1=real(f1);y1=imag(f1);
plot(x1,y1,'*')
```

plot on next pg.

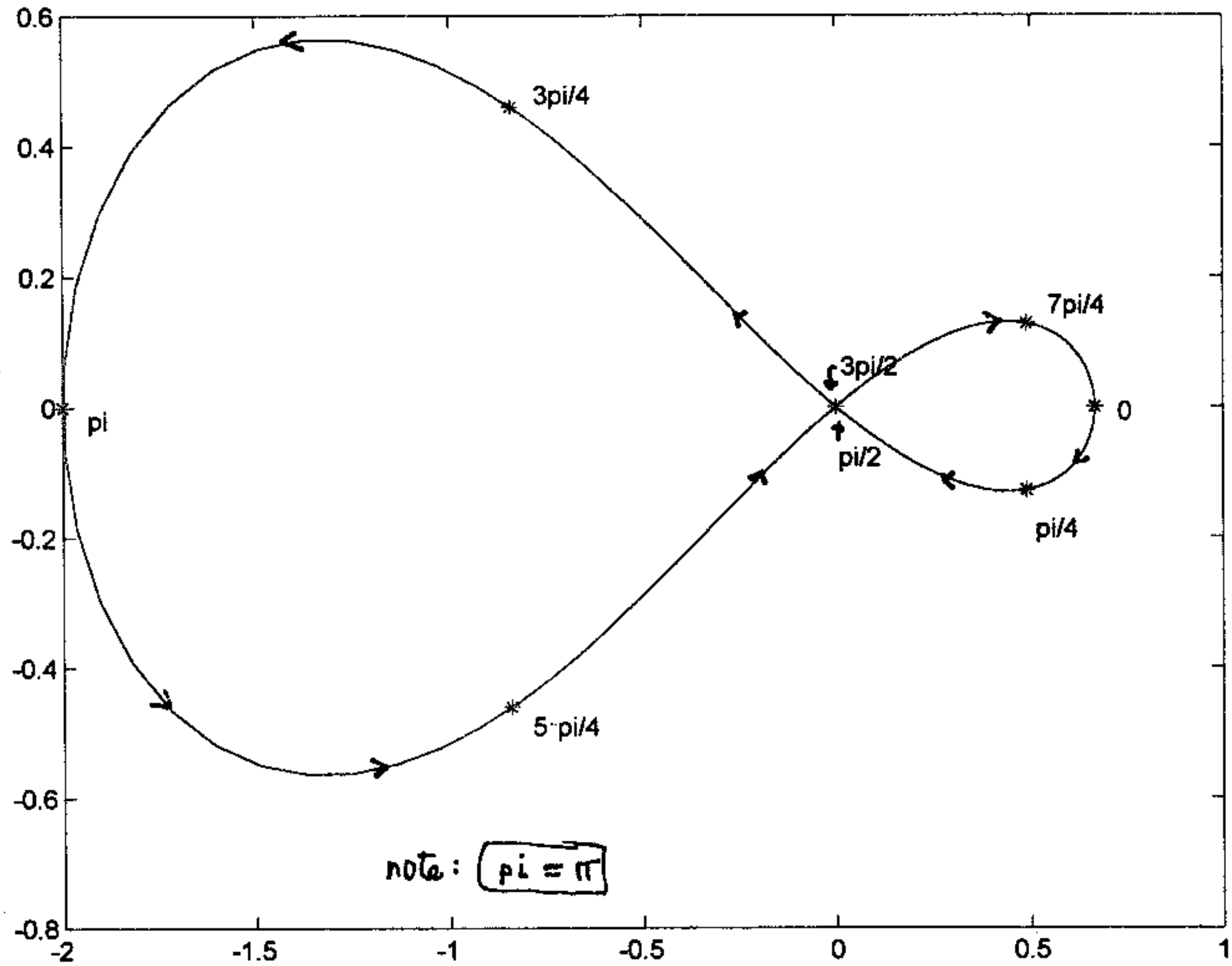
(e) $$f'(t) = \frac{\left[-\sin t - \frac{1}{2} i\right]}{\left[1 + \frac{1}{2}\left[\cos t + i \sin t\right]\right]^2}$$

```
% problem 20 sec 2.4,part(e)
t=linspace(0,2*pi,100);
f=(-sin(t)-.5*i)./(1+.5*(cos(t)+i*sin(t))).^2;
x=real(f);
y=imag(f);
plot(x,y);hold on
t1=linspace(0,2*pi,9);
f1=(-sin(t1)-.5*i)./(1+.5*(cos(t1)+i*sin(t1))).^2;
x1=real(f1);y1=imag(f1);
plot(x1,y1,'*');grid
```

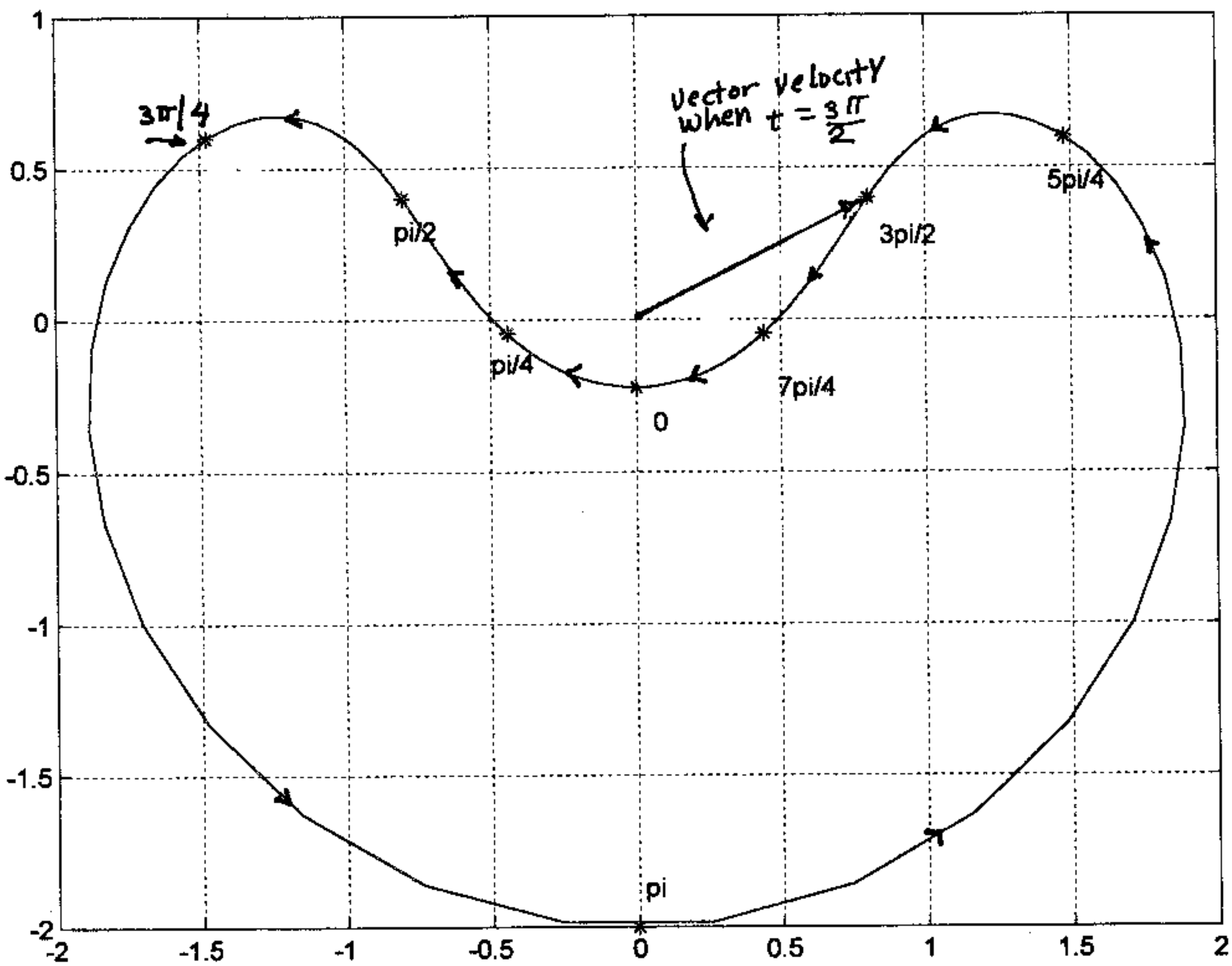
plot is on pg. after next.

20 (d)

continued



part d) Problem 20 section 2.4



problem 2a e section 2.4

Chap 2 page 36

21| sec 2.4,

$$\frac{\partial u}{\partial r} = \cos\theta, \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = 0 \quad \therefore \cos\theta = 0$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{-r\sin\theta}{r}, \quad \frac{\partial v}{\partial r} = 1$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad \therefore \sin\theta = 1$$

If $\cos\theta = 0$ and $\sin\theta = 1$, then $\theta = \frac{\pi}{2}$

Derivative exists only on the ray $\theta = \frac{\pi}{2}$, $0 < r < \infty$. This is not a domain. $\therefore$ function not analytic.

22| $u = r^4 \sin 4\theta$, $v = -r^4 \cos 4\theta$

$$\frac{\partial u}{\partial r} = 4r^3 \sin 4\theta, \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = 4r^3 \sin 4\theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{for all } r \neq 0$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 4r^3 \cos(4\theta) = -\frac{\partial v}{\partial r} \quad \text{for all } r, \theta$$

We have proved that

The function is analytic for all $r \neq 0$ [that is for all $z \neq 0$]. If $z=0$ the proof breaksdown

23)

$u = u(r(x,y), \theta(x,y))$

Thus $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\Big|_\theta \frac{\partial r}{\partial x}\Big|_y + \frac{\partial u}{\partial \theta}\Big|_r \frac{\partial \theta}{\partial x}\Big|_y$ [1]

$\frac{\partial u}{\partial y}$ (same as [1] but swap x and y

$\frac{\partial v}{\partial x}$ same as [1] but put v instead of u

$\frac{\partial v}{\partial y}$ same as [1] but put v instead of u, swap x and y.

(b) $r = \sqrt{x^2+y^2}$, $\frac{\partial r}{\partial x}\Big|_y = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\frac{\partial \theta}{\partial x}\Big|_y = \frac{1}{1+y^2/x^2} \frac{-y}{x^2} = \frac{-y}{x^2+y^2} =$

$-\frac{y}{\sqrt{x^2+y^2}} \frac{1}{\sqrt{x^2+y^2}} = -\frac{\sin\theta}{r}$

$\frac{\partial r}{\partial y}\Big|_x = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$, $\frac{\partial \theta}{\partial y}\Big|_x = \frac{x}{\sqrt{x^2+y^2}} \frac{1}{\sqrt{x^2+y^2}} = \frac{\cos\theta}{r}$

Use the preceding equations in [1]

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\Big|_\theta \frac{\partial r}{\partial y}\Big|_x + \frac{\partial u}{\partial \theta}\Big|_r \frac{\partial \theta}{\partial y}\Big|_x$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$ similarly:

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$

(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ from (2.3-10a)

Using result of part (b):

$\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$ [2]

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ from (2-3-10b)

$$\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial\theta}\sin\theta = -\frac{\partial u}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial u}{\partial\theta}\cos\theta \quad [3]$$

Multiply [3] by $\sin\theta$ and [2] by $\cos\theta$ and add [2], [3]

Subtract $\frac{\partial v}{\partial r}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial u}{\partial\theta}\sin\theta\cos\theta$

from each side of the result, you get:

$$\frac{\partial u}{\partial r}\cos^2\theta - \frac{1}{r}\frac{\partial v}{\partial\theta}\sin^2\theta = \frac{1}{r}\frac{\partial v}{\partial\theta}\cos^2\theta - \frac{\partial u}{\partial r}\sin^2\theta$$

or $\frac{\partial u}{\partial r}\left[\cos^2\theta+\sin^2\theta\right] = \frac{1}{r}\frac{\partial v}{\partial\theta}\left[\sin^2\theta+\cos^2\theta\right]$

or $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$ q.e.d. Now multiply [2] by $(-\sin\theta)$

and [3] by $(\cos\theta)$. Add the resulting eqns.

and add $\frac{\partial u}{\partial r}\sin\theta\cos\theta + \frac{1}{r}\sin\theta\cos\theta\frac{\partial v}{\partial\theta}$ to each side of the result. Get:

$$\frac{\partial v}{\partial r}\cos^2\theta + \frac{1}{r}\frac{\partial u}{\partial\theta}\sin^2\theta = -\frac{\partial v}{\partial r}\sin^2\theta - \frac{1}{r}\frac{\partial u}{\partial\theta}\cos^2\theta$$

$$\frac{\partial v}{\partial r}\left[\cos^2\theta+\sin^2\theta\right] = -\frac{1}{r}\frac{\partial u}{\partial\theta}\left[\cos^2\theta+\sin^2\theta\right)$$

or $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}$ q.e.d

(d) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial\theta}\sin\theta$

$+ i\left[\frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial\theta}\sin\theta\right]$. But $-\frac{1}{r}\frac{\partial u}{\partial\theta} = \frac{\partial v}{\partial r}$

and $-\frac{1}{r}\frac{\partial v}{\partial\theta} = -\frac{\partial u}{\partial r}$ (the polar form of C-R equations)

Hence $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \sin\theta\frac{\partial v}{\partial r}$

$+ i\left[\frac{\partial v}{\partial r}\cos\theta - \frac{\partial u}{\partial r}\sin\theta\right] = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\theta - i\sin\theta)$

q.e.d

To get the other form of $f'(z)$ put in the preceding expression for $f'(z)$ the following: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$

$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}$. Thus $f'(z) = \left(\frac{1}{r}\frac{\partial v}{\partial\theta} - \frac{i}{r}\frac{\partial u}{\partial\theta}\right)(\cos\theta - i\sin\theta)$

$= -\frac{i}{r}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right]\left[\cos\theta - i\sin\theta\right]$ q.e.d

Section 2.5

1) $\phi = x^2 - y^4$

$\partial\phi/\partial x = 2x, \quad \dfrac{\partial^2\phi}{\partial x^2} = 2$

$\dfrac{\partial\phi}{\partial y} = -4y^3, \quad \dfrac{\partial^2\phi}{\partial y^2} = -12y^2$

$2 - 12y^2 = 0 \qquad 12y^2 = 2, \quad y = \pm\sqrt{\tfrac{1}{6}}$

The equation is satisfied only on the lines $y = \pm\sqrt{1/6} \quad -\infty < x < \infty$

These sets of points are not a domain, thus the function is not harmonic.

2) $\phi = \sin(xy) \qquad \dfrac{\partial^2\phi}{\partial x^2} = -y^2\sin(xy)$

$\dfrac{\partial^2\phi}{\partial y^2} = -x^2\sin(xy) \therefore \; -\sin(xy)\,[x^2+y^2] = 0$

The preceding eqn is satisfied at the origin $(x=0, y=0)$ or on the hyperbolas $xy = n\pi \quad [n = 0, \pm1, \pm2 \ldots]$. These sets of points are not a domain. $\therefore$ func. not harmonic

3) $\phi = e^{ky}\sin(mx) \qquad \dfrac{\partial^2\phi}{\partial x^2} = -m^2 e^{ky}\sin(mx)$

$\dfrac{\partial^2\phi}{\partial y^2} = k^2 e^{ky}\sin(mx) \qquad \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$

$[k^2 - m^2]\,[e^{ky}\sin mx] = 0 \qquad \boxed{k = \pm m}$

4) $\phi = x^n - y^n \qquad \partial^2\phi/\partial x^2 = (n)(n-1)\,x^{n-2}$

$\dfrac{\partial^2\phi}{\partial y^2} = -(n)(n-1)\,y^{n-2} \qquad \boxed{n=0,\ n=1}$

$(n)(n-1)\,[x^{n-2} - y^{n-2}] = 0 \quad$ or $x^{n-2} = y^{n-2}$

$\boxed{n=2}$

section 2.5

5] $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$Im(1/z) = \frac{-y}{x^2+y^2} = \phi$

$\frac{\partial\phi}{\partial y} = -\left[\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right] = -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$

$\frac{\partial^2\phi}{\partial y^2} = -\left[\frac{-2y\,(x^2+y^2)^2 - (x^2-y^2)\,2[x^2+y^2]\,2y}{(x^2+y^2)^4}\right]$

$\frac{\partial^2\phi}{\partial y^2} = \frac{-2y^3+6x^2y^2}{(x^2+y^2)^3}$

similarly $\frac{\partial^2\phi}{\partial x^2} = \frac{2y^3-6xy^2}{(x^2+y^2)^3}$

thus

$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ if $z \neq 0$

6] $z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 - iy^3$

$Re(z^3) = x^3 - 3xy^2 = \phi$

$\partial\phi/\partial x = 3x^2 - 3y^2$, $\frac{\partial^2\phi}{\partial x^2} = 6x$

$\frac{\partial\phi}{\partial y} = -6xy$, $\frac{\partial^2\phi}{\partial y^2} = -6x$ $\quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

all x, y.

7] $\phi = \cos x\,[e^y + e^{ky}]$

$\frac{\partial^2\phi}{\partial x^2} = -\cos x\,[e^y + e^{ky}]$

$\frac{\partial^2\phi}{\partial y^2} = \cos x\,[e^y + k^2e^{ky}]$

$-\cos x\,[e^y + e^{ky}] + \cos x\,[e^y + k^2e^{ky}] = 0 \quad k = \pm 1$

8] $\phi = g(x)\left[e^{2y} - e^{-2y}\right]$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{d^2 g}{dx^2}\left[e^{2y} - e^{-2y}\right]$$

$$\frac{\partial^2 \phi}{\partial y^2} = g(x) \quad 4\left[e^{2y} - e^{-2y}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \left[4g(x) + \frac{d^2 g}{dx^2}\right]\left[e^{2y} - e^{-2y}\right] = 0$$

$$\frac{d^2 g}{dx^2} + 4g(x) = 0 \qquad g(x) = A\cos 2x + B\sin 2x$$

$A = 0$ since $g(0) = 0$

$g'(x) = 2B\cos(2x)$

$g'(0) = 1 \quad \Rightarrow B = 1/2$

$\therefore\ g(x) = \frac{1}{2}\sin 2x$

9] a) $\phi = x^3 y - y^3 x + y^2 - x^2 + x$

$$\frac{\partial^2 \phi}{\partial x^2} = 6xy - 2, \quad \frac{\partial^2 \phi}{\partial y^2} = -6xy + 2$$

The sum is zero.

b) Take $\phi = u$, $\frac{\partial u}{\partial x} = 3x^2 y - y^3 - 2x + 1 = \frac{\partial v}{\partial y}$

$$v = \frac{3}{2}x^2 y^2 - \frac{y^4}{4} - 2xy + y + c(x)$$

$$\frac{\partial v}{\partial x} = 3xy^2 - 2y + \frac{dc}{dx} = -\frac{\partial u}{\partial y} = -x^3 + 3xy^2 - 2y$$

$dc/dx = -x^3$, $c = -x^4/4 + D$

$$v = \frac{3}{2}x^2 y^2 - \frac{y^4}{4} - 2xy + y - \frac{x^4}{4} + D$$

D is a constant.

9 cont'd

c) $v = x^3 y - y^3 x + y^2 - x^2 + x$

$$\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y} = x^3 - 3y^2 x + 2y$$

$$w = \frac{x^4}{4} - \frac{3}{2} x^2 y^2 + 2xy + c(y)$$

Now use $\dfrac{\partial w}{\partial y} = \dfrac{-\partial v}{\partial x}$

$$-3x^2 y + 2x + \frac{dc}{dy} = -3x^2 y + y^3 + 2x - 1$$

$$\frac{dc}{dy} = y^3 - 1, \qquad c = \frac{y^4}{4} - y + D$$

$$\therefore\ w = \frac{x^4}{4} - \frac{3}{2} x^2 y^2 + 2xy + \frac{y^4}{4} - y + D'$$

The answers to b) and c) are negatives of each other - if you neglect the arbitrary constants D and D'.

d) $\phi + iv$ is analytic

Multiply it by i

$\therefore\ -v + i\phi$ is analytic

Now by assumption $w + i\phi$ is analytic. $\therefore\ w = -v +$ constant

If we neglect the constant, $w = -v$ as req'd.

This is confirmed in parts b) and c).

sec 2.5 cont'd

10. $f(z) = u + iv$ is analytic

$-if(z) = v - iu$ is analytic.

$g(z) = v + iw$ is analytic

$g(z) - if(z)$ is analytic [sum of analyt. funcs]

$g(z) - if(z) = 2v$ is real, and analytic.

$\therefore$ by Exercise 15, sec 2.4

v is constant.

$g(z) + if(z) = 2iw$ is imag. and analytic

$\therefore$ by excercise 15, sec 2.4

w is constant.

Thus if $u + iv$ and $v + iw$ are both analytic, then w and v are constant.

11. $u = e^x \cos y + e^y \cos x + xy$, $\frac{\partial u}{\partial x} = e^x \cos y - e^y \sin x + y = \frac{\partial v}{\partial y}$

$v = e^x \sin y - e^y \sin x + y^2/2 + c(x)$, $\frac{\partial v}{\partial x} = e^x \sin y - e^y \cos x + \frac{dc}{dx}$

$= -\frac{\partial u}{\partial y} = e^x \sin y - e^y \cos x - x$. $\frac{dc}{dx} = -x$, $c = -\frac{x^2}{2} + D$

$$v = e^x \sin y - e^y \sin x + \frac{y^2}{2} - \frac{x^2}{2} + D$$

12. $u = \tan^{-1}\left(\frac{x}{y}\right)$, $\frac{\partial u}{\partial x} = \frac{1/y}{1 + x^2/y^2} = \frac{y}{y^2 + x^2} = \frac{\partial v}{\partial y}$

$v = \frac{1}{2} \operatorname{Log}(y^2 + x^2) + c(x)$. $\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + \frac{dc}{dx} = -\frac{\partial u}{\partial y}$

$-\frac{\partial u}{\partial y} = \frac{x/y^2}{1 + x^2/y^2} = \frac{x}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{dc}{dx}$, $\frac{dc}{dx} = 0$, $c = D$

$\therefore$ $v = \frac{1}{2} \operatorname{Log}(y^2 + x^2) + D$

chap 2, 13] Sec 2.5 cont'd

let $\phi = u+v$, show $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$$\frac{\partial^2}{\partial x^2}(u+v) + \frac{\partial^2}{\partial y^2}(u+v) = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v$$

$= 0$ since both u and v are harmonic.

let $\phi = uv$, $\frac{\partial^2}{\partial x^2}\phi = \frac{\partial^2}{\partial x^2}(uv) = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}v + \frac{\partial v}{\partial x}u\right]$

$= \frac{\partial^2 u}{\partial x^2}v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}$. Similarly:

$$\frac{\partial^2}{\partial y^2}(uv) = \frac{\partial^2 u}{\partial y^2}v + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2}$$

Now $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right][uv] = v\underbrace{\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]}_{=0} +$

$u\underbrace{\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]}_{=0} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} =$

$2\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$. The preceding is not zero in general.

$e^{u+v} = e^u e^v = \phi$, $\frac{\partial \phi}{\partial x} = e^{u+v}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right]$

$$\frac{\partial^2 \phi}{\partial x^2} = e^{u+v}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right]^2 + e^{u+v}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right]$$

$$\frac{\partial^2 \phi}{\partial y^2} = e^{u+v}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right]^2 + e^{u+v}\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

Sum the above, and use $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = e^{u+v}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)^2\right]$$

The preceding is in general $\neq 0$.

So e^{u+v} is not harmonic.

14] Refer to problem 13, We have that if both u and v are harmonic, then $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right](uv) = 2\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$

Now put $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in the bracketed expression. You will get zero.

15] $\phi = e^u \cos v$, $\frac{\partial \phi}{\partial x} = \phi \frac{\partial u}{\partial x} - e^u \sin v \frac{\partial v}{\partial x}$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial x}\frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2} - e^u \sin v \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - e^u \cos v \left(\frac{\partial v}{\partial x}\right)^2 - e^u \sin v \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial x}\frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2} - e^u \sin v \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \phi \left(\frac{\partial v}{\partial x}\right)^2 - e^u \sin v \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial x}\frac{\partial u}{\partial x} + \phi\left[\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial v}{\partial x}\right)^2\right] - e^u \sin v \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right]$$

here

similarly

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial y}\frac{\partial u}{\partial y} + \phi\left[\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial v}{\partial y}\right)^2\right] - e^u \sin v \left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}\right]$$

Add the above 2 eqns, use fact that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and C-R eqns. put $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in here

Get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial u}{\partial y} - \phi\left[\left[\frac{\partial v}{\partial x}\right]^2 + \left[\frac{\partial v}{\partial y}\right]^2\right]$$

use

Now $\frac{\partial \phi}{\partial x} = \phi \frac{\partial u}{\partial x} - e^u \sin v \frac{\partial v}{\partial x}$,

$\frac{\partial \phi}{\partial y} = \phi \frac{\partial u}{\partial y} - e^u \sin v \frac{\partial v}{\partial y}$ use

Thus
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \phi\left(\frac{\partial u}{\partial x}\right)^2 - e^u \sin v \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \phi\left[\frac{\partial u}{\partial y}\right]^2 - e^u \sin v \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - \phi\left(\frac{\partial v}{\partial x}\right)^2 - \phi\left(\frac{\partial v}{\partial y}\right)^2$$

$$= -e^u \sin v \left[\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right] = -e^u \sin v \left[-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]$$

Use C-R eqns

$= 0$ q.e.d.

16 $\phi = \sin u \cosh v$, $\dfrac{\partial\phi}{\partial x} = \cos u \cosh v \dfrac{\partial u}{\partial x} + \sin u \sinh v \dfrac{\partial v}{\partial x}$

$$\frac{\partial^2\phi}{\partial x^2} = \cosh v \sin u\left[\left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right] + \cos u \cosh v \frac{\partial^2 u}{\partial x^2}$$

$+ \sin u \sinh v \dfrac{\partial^2 v}{\partial x^2} + 2\cos u \sinh v \dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x}$. Similarly

$$\frac{\partial^2\phi}{\partial y^2} = \cosh v \sin u\left[\left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right] + \cos u \cosh v \frac{\partial^2 u}{\partial y^2}$$

$+ \sin u \sinh v \dfrac{\partial^2 v}{\partial y^2} + 2\cos u \sinh v \dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}$

Add: $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = \cosh v \sin u\left[\left(\dfrac{\partial v}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2\right]$

$+ \cos u \cosh v\left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right] + \sin u \sinh v\left[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right]$

$+ 2\cos u \sinh v\left[\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}\right]$. Now in preceding

use $-\dfrac{\partial u}{\partial y}$ in place of $\dfrac{\partial v}{\partial x}$, and $\dfrac{\partial u}{\partial x}$ in place of $\dfrac{\partial v}{\partial y}$.

Also $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$. Get

$\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ as required.

17 a) $(x+iy)^2 = u+iv$, $u = x^2-y^2$, $v = 2xy$

$x^2-y^2=1$, $xy=1$, (b) $x^2-y^2=1$, $y=1/x$, $x^2-\frac{1}{x^2}=1$

$x^4-x^2-1=0$, $x^2 = \dfrac{1\pm\sqrt{1+4}}{2}$, $x^2>0$, $x^2=\dfrac{1+\sqrt5}{2}$,

$x=\sqrt{\dfrac{1+\sqrt5}{2}}$, $y^2=x^2-1 = \dfrac{\sqrt5-1}{2}$, $y=\sqrt{\dfrac{\sqrt5-1}{2}}$

c) $x^2-y^2=1$, $2x\,dx - 2y\,dy = 0$

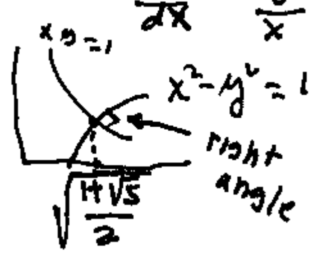
$$\frac{dx}{dy} = \frac{x}{y} = \sqrt{\frac{1+\sqrt5}{\sqrt5-1}} \doteq 1.62$$

curve $xy=1$

$x\,dx + y\,dy = 0$

$\dfrac{dy}{dx} = -\dfrac{y}{x} \doteq -\dfrac{1}{1.62}$

slopes are neg. reciprocals.



sec 2.5

18 a) $u = e^x \cos y$, $v = e^x \sin y$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y = \partial v/\partial y \\ -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} = e^x \sin y \end{aligned}\right] \text{true, all } x, y$$

```
k=[1/2  1]
% for sec 2.5 prob.18. parts b), c)

for m=1:2
   x=linspace(0,pi/2,1000);
y=acos(k(m)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]);hold on
end
for m=1:2
x=linspace(0,pi/2,1000);
y=asin(k(m)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]);hold on
end
grid
```

Note: for part (b)

$u = e^x \cos y = 1$, $y = \cos^{-1}(e^{-x})$

If $u = 1/2$, get in a similar way

$y = \cos^{-1}[(1/2)e^{-x}]$

for part (c)

$v = e^x \sin y$, If $v = 1$, $y = \sin^{-1}(e^{-x})$

If $v = 1/2$, get in a similar way

$y = \sin^{-1}(e^{-x})$

chap page 48

18

Sec 2.5

d) $e^x \cos y = 1$

$e^x \sin y = 1/2$ divide 2nd by 1st.

$\tan y = 1/2$, $y = \arctan 1/2 \doteq .4636$

$\sin y = .4472$ $e^x = \frac{1}{(2)(.4472)}$

$x = \log\left[\frac{.5}{.4472}\right] = .1116$ $\boxed{\text{plot on next pg.}}$

e) $e^x \cos y = 1$

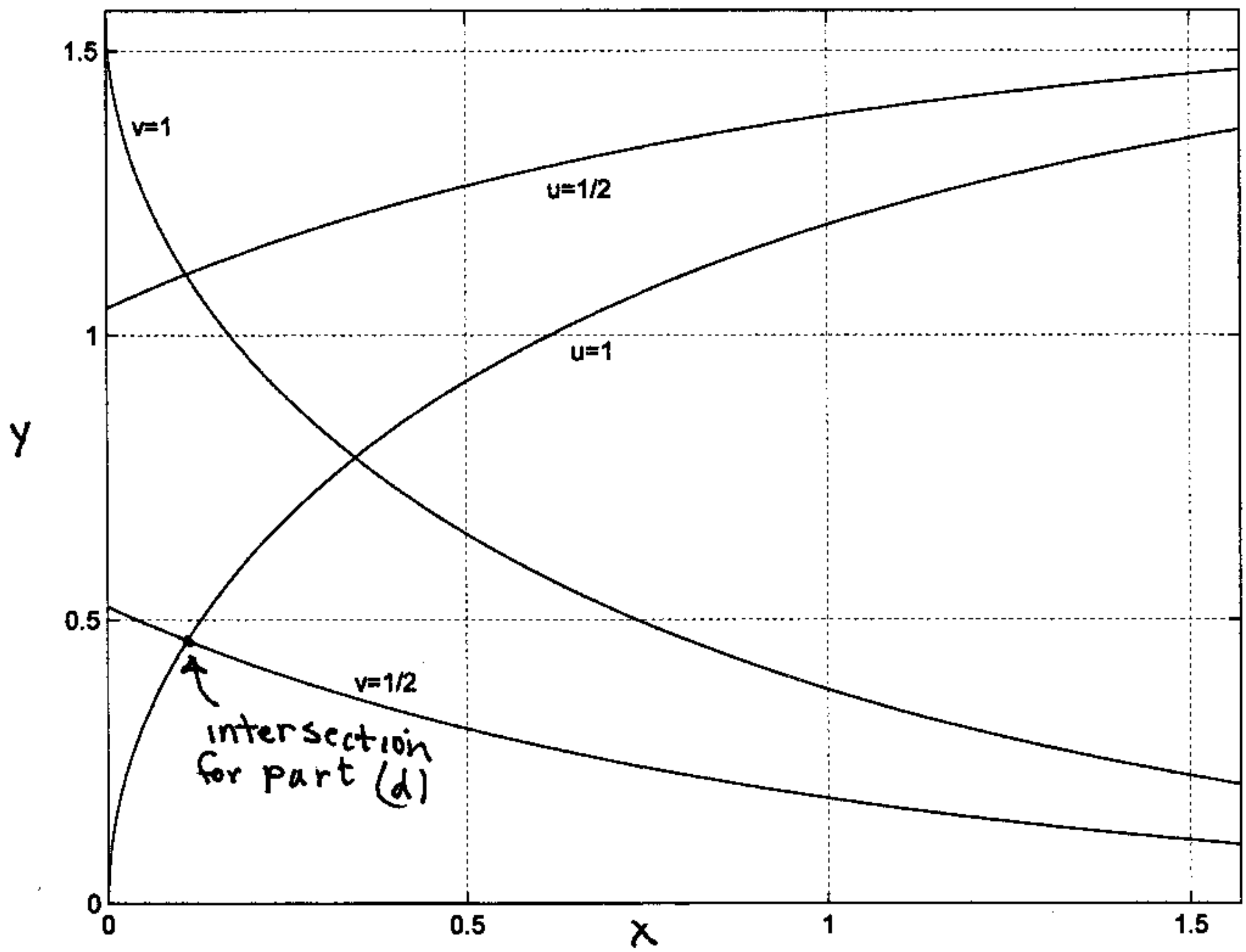
$e^x\, dx \cos y - e^x \sin y\, dy = 0$

$dy/dx = \cot y = \boxed{2}$ slope of $u = 1$

$e^x \sin y = 1/2$ $e^x \cos y\, dy + e^x \sin y\, dx = 0$

$\frac{dy}{dx} = -\tan y = \boxed{-1/2}$ slope of $v = 1/2$

These are neg. recips. of each other.



for problem 18(d), sec 2.5

Chap 2 page 50

sec 2.5

19] a) $z^3 = (x+iy)^3 = x^3 + i3x^2y - 3xy^2 - iy^3$

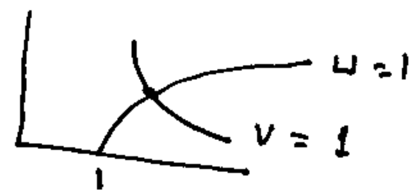
$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$

$u = 1, \quad x^3 - 3xy^2 = 1$

$y^2 = \dfrac{x^3-1}{3x}$ eqn. of $u=1$ curve

If $v=1, \quad 3x^2y - y^3 = 1$

$x^2 = \dfrac{y^3+1}{3y}$



b) $f(z) = z^3, \quad z = r\,cis(\theta), \quad z^3 = r^3[\cos 3\theta + i\sin 3\theta]$

$u + iv = f(z), \; u = r^3\cos 3\theta, \; v = r^3 \sin 3\theta, \quad r^3\cos 3\theta = 1$

$r^3 \sin 3\theta = 1$ dividing these: $\tan 3\theta = 1, \quad \theta = 15° = \frac{\pi}{12}$

$r^3 \cos 45° = 1, \; r^3 = \sqrt{2}, \; r = \sqrt[6]{2} \approx 1.122, \; x = r\cos\theta$

$\underline{x = 1.122\cos 15° = 1.083}, \; \underline{y = 1.122 \sin 15° = .290}$

c) $u = x^3 - 3xy^2 = 1, \; v = 3x^2y - y^3$

$du = 0 = (3x^2 - 3y^2)dx - 6xy\,dy, \quad \dfrac{dy}{dx} = \dfrac{3x^2-3y^2}{6xy} = \dfrac{x^2-y^2}{2xy}$

At $x = 1.083, y = .290, \; \boxed{\dfrac{dy}{dx} = 1.73}, \; dv = 0 = 6xy\,dx + (3x^2-3y^2)dy$

$\dfrac{dy}{dx} = \dfrac{-6xy}{3x^2-3y^2} = \dfrac{-2xy}{x^2-y^2} = \dfrac{-1}{1.73} = \underline{\underline{-.58}}$ at intersection.

sec 2.5

20

a) Begin with $\frac{\partial u}{\partial r} = 1/r \frac{\partial v}{\partial \theta}$ [1] and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ [2]

Take $\frac{\partial}{\partial r}$ of [1] get $\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial v}{\partial \theta}$ [3]

Take $\frac{\partial}{\partial \theta}$ of [2] and divide both sides by r, get

get $\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}$ [4] Add [3] and [4] assume $\frac{\partial}{\partial r}\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\frac{\partial v}{\partial r}$

get $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta}$ [5]. From [1] $-\frac{1}{r}\frac{\partial u}{\partial r} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta}$

Use this on right side of [5] $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r}$

or $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$ q.e.d

Now take $\frac{\partial}{\partial r}$ of [2]. Get $\frac{\partial^2 v}{\partial r^2} = \frac{1}{r^2}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial u}{\partial \theta}$ [6]. Now

take $\frac{\partial}{\partial \theta}$ of [1]. Get $\frac{\partial^2 u}{\partial \theta \partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}$ or

$\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{r}\frac{\partial^2 u}{\partial \theta \partial r}$ [7]. Add [6] and [7]

$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial u}{\partial \theta}$ From [2] $\frac{1}{r^2}\frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial r}$ use here

Thus $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + 1/r\frac{\partial v}{\partial r} = 0$ q.e.d

b) next page

20 (b)

continued

b) $\frac{\partial u}{\partial r} = 2r\cos(2\theta)$, $\frac{\partial^2 u}{\partial r^2} = 2\cos(2\theta)$,

$\frac{\partial u}{\partial \theta} = -2r^2 \sin(2\theta)$, $\frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos(2\theta)$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 2\cos(2\theta) - 4\cos(2\theta) + 2\cos 2\theta$

$= 0$ (q.e.d)

c) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $u = r^2\cos(2\theta)$, $2r\cos(2\theta) = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$v(\theta) = r^2 \sin(2\theta) + c(r)$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$2r\sin(2\theta) + \frac{dc}{dr} = \frac{2r^2}{r}\sin(2\theta)$. $\frac{dc}{dr} = 0$, $c = d$ (a constant)

thus $\underline{v = r^2\sin(2\theta) + \text{constant}}$. Check Laplace's

Eqn. $v = r^2\sin(2\theta) + \text{const}$. $\frac{\partial v}{\partial r} = 2r\sin(2\theta)$

$\frac{\partial^2 v}{\partial r^2} = 2\sin(2\theta)$, $\frac{\partial v}{\partial \theta} = r^2\cos(2\theta)$, $\frac{\partial^2 v}{\partial \theta^2} = -4r^2\sin 2\theta$

$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 2\sin(2\theta) - 4\sin(2\theta) + 2\sin(2\theta)$

$= 0$ q.e.d

Sec 2.6

1 (a) $Q_x = -k \frac{\partial \phi}{\partial x} = 3 = -.1 \frac{\partial \phi}{\partial x}$ $\therefore$ $\phi = -30x + c(y)$

$-4 = -.1 \frac{\partial \phi}{\partial y}$. Thus $\frac{dc}{dy} = 40$, $c = 40y + d$

$\phi = -30x + 40y$ + constant. const $= 0$ $\underline{\phi = -30x + 40y}$ degrees

(b) $\phi = -30x + 40y$ $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, $-30 = \frac{\partial \psi}{\partial y}$,

$\psi = -30y + c(x)$, $\frac{\partial \phi}{\partial y} = -\frac{dc}{dx}$, $40 = -\frac{dc}{dx}$, $c = -40x + d$

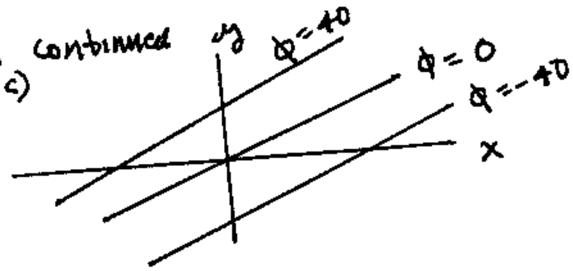
$\therefore \psi = -30y - 40x$ degrees (constant is zero)

(c) $\phi = 0$, $-30x + 40y = 0$, $y = (3/4)x$; $\phi = 40 = -30x + 40y$

$y = 1 + (3/4)x$; $\phi = -40$, $y = -1 + \frac{3}{4}x$

1) continued

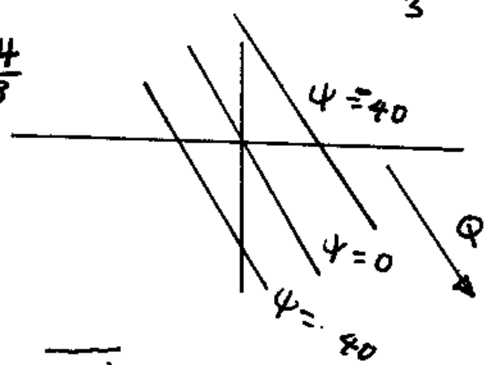
c)



d) $\psi = 0, = -30y - 40x, \quad x = -\frac{3}{4}y, \quad y = -\frac{4}{3}x$

$\psi = 40 = -30y - 40x, \quad y = -\frac{4}{3}x - \frac{4}{3}; \quad \psi = -40,$

$y = -\frac{4}{3}x + \frac{4}{3}$



2 a) $v = \overline{\left(\frac{d\Phi}{dz}\right)} = \overline{\frac{1}{z^2}}$. But $\frac{-1}{z^2} = \frac{-1}{x^2 - y^2 + i2xy}$

$= \frac{-(x^2 - y^2 - i2xy)}{(x^2 - y^2)^2 + 4x^2y^2} = \frac{-(x^2 - y^2 - i2xy)}{x^4 - 2x^2y^2 + y^4 + 4x^2y^2} =$

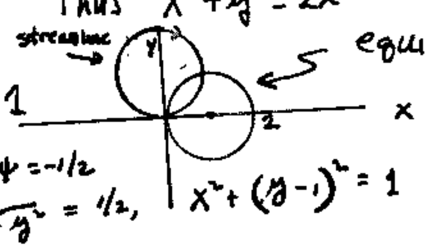
$\frac{-(x^2 - y^2 - i2xy)}{(x^2 + y^2)^2}$. Thus, $v = \frac{-(x^2 - y^2 + i2xy)}{(x^2 + y^2)^2} = V_x + iV_y$

If $x = 1, y = 1$, $\underline{\underline{V_x = 0}}$, $V_y = \frac{-2}{4} = \underline{\underline{\frac{-1}{2} = V_y}}$

b) $\phi = \mathrm{Re}\left[\frac{1}{z}\right] = \frac{x}{x^2 + y^2}$, $V_x = \frac{\partial\phi}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$

$V_y = \frac{\partial\phi}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$. Putting $x = 1, y = 1$, get $V_x = 0$, $V_y = \frac{-1}{2}$

c) $\phi = \left.\frac{x}{x^2 + y^2}\right|_{1,1} = 1/2$ Thus $x^2 + y^2 = 2x$



equipot.

$x^2 - 2x + y^2 = 0, \quad (x-1)^2 + y^2 = 1$

d) $\psi = \mathrm{Im}(1/z) = \frac{-y}{x^2 + y^2}$, at 1,1, $\psi = -1/2$

Thus $\frac{y}{x^2 + y^2} = 1/2$, $x^2 + (y-1)^2 = 1$ ← equation

3] a) $\Phi(z) = \underbrace{e^x \cos y}_{\phi} + i \underbrace{e^x \sin y}_{\psi}$, $\frac{d\Phi}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$

$= e^x \cos y + i e^x \sin y$. $\underline{e} = -\overline{\left(\frac{d\Phi}{dz}\right)} =$

$-[e^x \cos y - i e^x \sin y]$. Thus $\underline{e}$ at $1, 1/2$ $= -[\underbrace{e \cos\frac{1}{2} - ie \sin\frac{1}{2}}_{e\,\mathrm{cis}(-1/2)}]$

$= E_x + iE_y = -2.39 + i\,1.30$.

b) $\phi = e^x \cos y$, $E_x = -\frac{\partial\phi}{\partial x} = -e^x \cos y = -e \cos\frac{1}{2} = E_x$

$\vec{e} = E_x + iE_y$

$E_y = -\frac{\partial\phi}{\partial y} = e^x \sin y = e \sin 1/2 = E_y$ $= -e\,\mathrm{cis}(-1/2)$ as in (a)

c) $D_x = -8.85 \times 10^{-12} e \cos\frac{1}{2}$, $D_y = 8.85 \times 10^{-12} e \sin\frac{1}{2}$

$\underline{D_x = -21. \cdot 10^{-12}}$, $\underline{D_y = 11.5 \times 10^{-12}}$

d) $\phi = e^x \cos y = \boxed{e \cos\frac{1}{2}}$

$e^x \cos y = e \cos\frac{1}{2}$

$y = \cos^{-1}\left[e \cos\frac{1}{2} e^{-x}\right]$

see attached plot

Suppose $e \cos\frac{1}{2} e^{-x} = 1$

$x = \log\left[e \cos\frac{1}{2}\right] = .8694$

on our plot take $x \geq .8694$ to avoid taking $\cos^{-1}$ of a number > 1

e) $\psi = e^x \sin y = e \sin 1/2$,

$y = \sin^{-1}\left[e \sin\frac{1}{2} e^{-x}\right]$

see attached plot

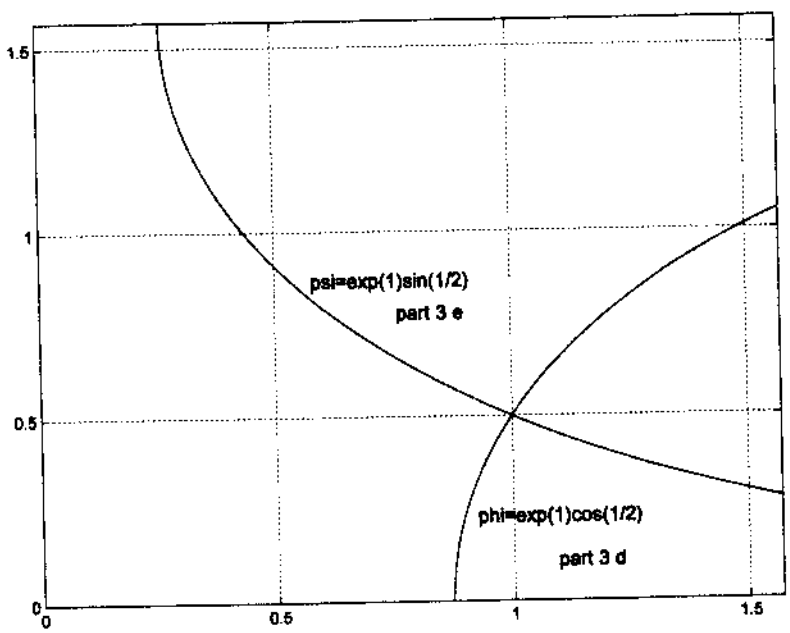
suppose $e \sin\frac{1}{2} e^{-x} = 1$

$x = \log\left[e \sin\frac{1}{2}\right] = .2684$

for plot, take $x \geq .2684$

to avoid taking $\sin^{-1}$ of a number > 1

```
k=[1/2  1]
% for sec 2.6prob3 (d,e)
%part d
   x=linspace(0,pi/2,1000);
y=acos(exp(1)*cos(1/2)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]);hold on
%part e
y=asin(exp(1)*sin(1/2)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]);hold on
end
grid
```



4 (a) We require that D_x, D_y satisfy Eqn (2.6-3)

If $\underline{d} = y + ix$, $D_x = y$, $D_y = x$, $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0$

If $\underline{d} = x + iy$, $D_x = x$, $D_y = y$, $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 2 \neq 0$

(b) $d = y + ix$, $D_x = y$, $D_y = x$, $-\epsilon \frac{\partial \phi}{\partial x} = D_x = y$

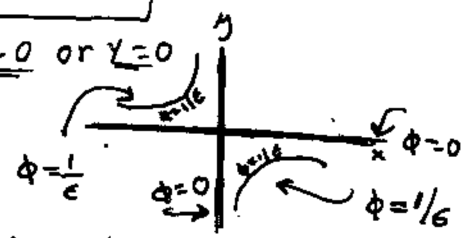
$\phi = -\frac{1}{\epsilon} xy + c(y)$. Recall $-\epsilon \frac{\partial \phi}{\partial y} = D_y = x$

Thus $x - \epsilon \frac{d\,c(y)}{dy} = x$, $\frac{dc}{dy} = 0$, c = const.

$\phi = -\frac{1}{\epsilon} xy + \underbrace{\text{const}}_{\to 0} = \boxed{-\frac{1}{\epsilon} xy = \phi}$

$\phi = 0$, $-\frac{1}{\epsilon} xy = 0$, $\underline{x = 0}$ or $\underline{y = 0}$

$\phi = \frac{1}{\epsilon} = -\frac{1}{\epsilon} xy$, $xy = -1$



c) $\phi = -\frac{1}{\epsilon} xy$, $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, $-\frac{y}{\epsilon} = \frac{\partial \psi}{\partial y}$, $\psi = -\frac{y^2}{2\epsilon} + c(x)$

$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$, $\frac{dc}{dx} = \frac{x}{\epsilon}$, $c = \frac{x^2}{2\epsilon}$ + const.

$\psi = \frac{x^2 - y^2}{2\epsilon}$ + const. Const. = 0 $[\psi(0,0) = 0]$. $\psi = \frac{x^2 - y^2}{2\epsilon}$

d) $\Phi = \phi + i\psi = -\frac{1}{\epsilon}\left[xy - i\left[\frac{x^2 - y^2}{2\epsilon}\right]\right] = \frac{i}{2\epsilon}\left[(x^2 - y^2) + i\,2xy\right]$

$= \boxed{\frac{i}{2\epsilon} z^2 = \Phi(z)}$

e) $\underline{e} = \underline{d}/\epsilon$ $\quad E_x = \frac{D_x}{\epsilon} = \frac{y}{\epsilon}\Big|_1 = \frac{1}{\epsilon}$,

$E_y = \frac{D_y}{\epsilon} = \frac{x}{\epsilon}\Big|_1 = \frac{1}{\epsilon}$. 2nd method $E_x + iE_y = -\overline{\left(\frac{d\Phi}{dz}\right)} = \frac{i\bar{z}}{\epsilon} =$

$\frac{i}{\epsilon}(x - iy) = \frac{y + ix}{\epsilon} = \frac{1+i}{\epsilon}$ at (1,1). Thus $E_x = \frac{1}{\epsilon} = E_y$

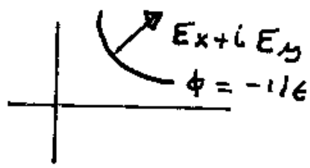
Third method: $\phi = -\frac{1}{\epsilon} xy$, $E_x = -\frac{\partial \phi}{\partial x} = \frac{y}{\epsilon}$, $E_y = -\frac{\partial \phi}{\partial y} = \frac{x}{\epsilon}$

$E_x = \frac{1}{\epsilon}$, $E_y = \frac{1}{\epsilon}$

Sec 2.6 continued

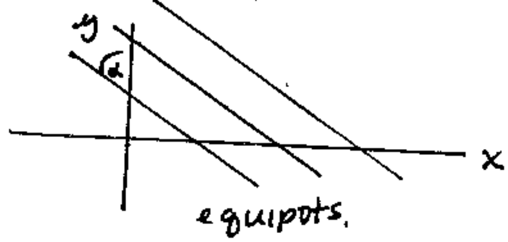
4(e) continued

Equipotential, $\phi = -\frac{1}{\epsilon} xy = -\frac{1}{\epsilon}$ at $(1,1)$. Thus $xy = 1$



5 (a) $\Phi = (\cos\alpha - i\sin\alpha)(x+iy) = x\cos\alpha + y\sin\alpha$
$+ i[y\cos\alpha - x\sin\alpha] = \phi + i\psi$. Thus $\phi = x\cos\alpha + y\sin\alpha$

Equipotentials $\quad x\cos\alpha + y\sin\alpha = \text{const.}$,
$dx\cos\alpha + dy\sin\alpha = 0 \qquad$ slope: $\frac{dy}{dx} = -\cot\alpha$

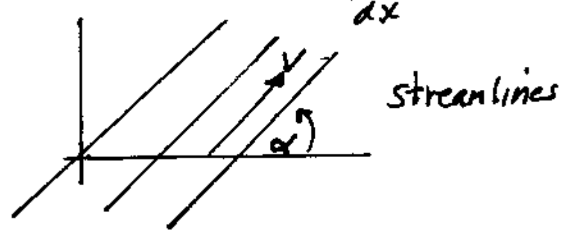


5(b) $\psi = y\cos\alpha - x\sin\alpha$

streamlines:

$y\cos\alpha - x\sin\alpha = \text{const.}$

slope $\frac{dy}{dx} = \tan\alpha$



5(c)

$\underline{v} = V_x + i V_y =$

$\overline{\left(\frac{d\Phi}{dz}\right)} = \cos\alpha + i\sin\alpha$

The vector makes an angle α with positive x axis.