

Chapter 2: Functions, Limits, and Continuity

Section 2.1: Functions

- function, Image= $\{0, 1, 4, 9, 16, 25, 36, \dots\}$.
- function, Image= The grades of the students.
- not a function; for $x = 1$ there are two values of $y \in [-2, 2]$ such that $x^2 + y^2 = 4$.
- function, Image= $\{x_1, x_2, x_3, \dots\}$.
- $[0, \frac{5\sqrt{10}}{2}]$
 - $h(0) = 1000, h(1) = 984, h(3) = 856, h(t+1) = -16(t+1)^2 + 1000$.
 - $h(t+1) - h(t)$ represents the distance covered by the falling object between t and $t+1$ seconds.
 - False. Corresponds to $h(t+1) = h(t) - h(1)$, while $h(t+1) = h(t) - 32t - 16 \neq h(t) - h(1)$.
- A. Graphing a race.**
 - B, about 2 hours 48 minutes;
 - A, the line has the greatest slope;
 - had a break;
 - about 25 km, all the riders slowed down;
 - 3 hours 30 minutes.

There are three functions, f_A, f_B, f_C , the distances travelled by the riders A, B, C, respectively.

f_A : Domain= $[0, 3]$, Image= $[0, 50]$,

f_B : Domain= $[0, 3]$, Image= $[0, 50]$,

f_C : Domain= $[0, 3]$, Image= $[0, 32.5]$.

B. Filling in temperatures.

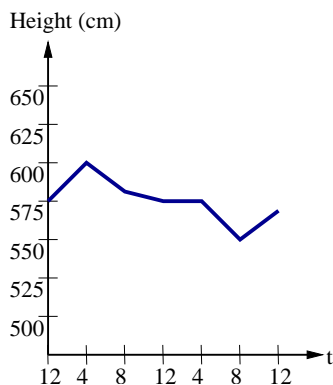
- The function is the height of the river in centimeters

Day 1		Day 2		Day 3		Day	
Time	Height	Time	Height	Time	Height	Time	Height
4 a.m.	510 cm	4 a.m.	585 cm	4 a.m.	580 cm	4 a.m.	635 cm
8 a.m.	510 cm	8 a.m.	585 cm	8 a.m.	605 cm	8 a.m.	635 cm
12 p.m.	510 cm	12 p.m.	580 cm	12 p.m.	620 cm	12 p.m.	625 cm
4 p.m.	510 cm	4 p.m.	575 cm	4 p.m.	620 cm	4 p.m.	600 cm
8 p.m.	550 cm	8 p.m.	573 cm	8 p.m.	630 cm	8 p.m.	580 cm
12 a.m.	575 cm	12 a.m.	570 cm	12 a.m.	635 cm	12 a.m.	580 cm

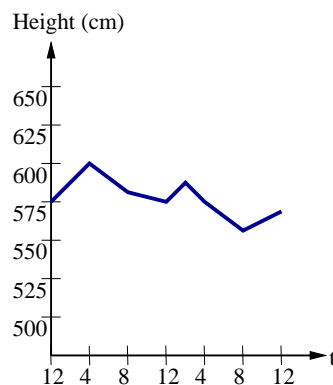
Domain=The 4 days time period.

Image=[510, 635].

2.a.



b.



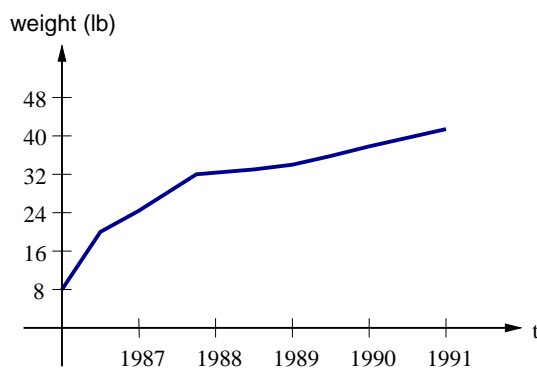
C. Graphs over a large time period.

1.a. 18 pounds;

b. April;

2. The function is Tameeka's weight in pounds. Domain=The 5 years time period (1987, Jan – 1991, Dec).

Image=[8, 41].



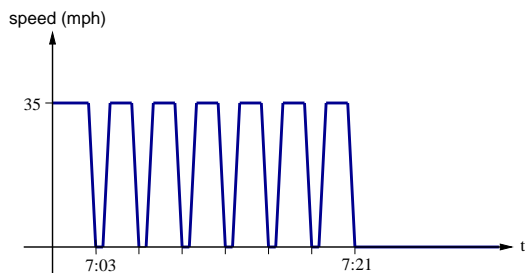
3.a. Tameeka's weight began to change slower and more uniform;

b. no;

c. the first year.

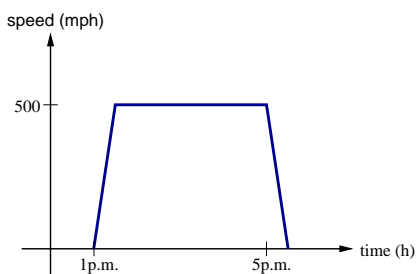
D. Graphs depicting speed.

1.



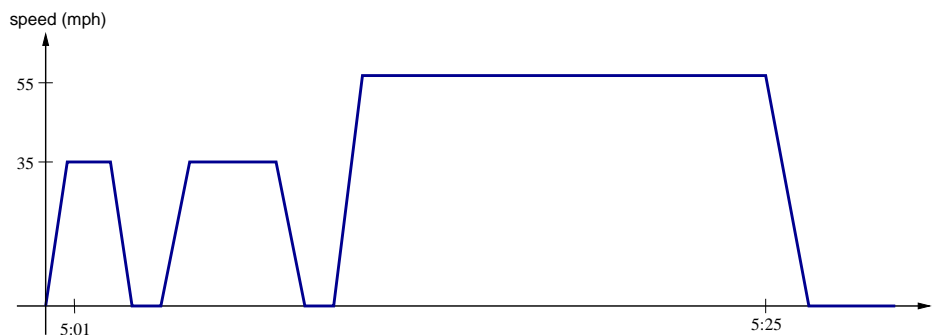
The function is the speed of the train in mph.
 Domain=The time period from 7 a.m. to 12 : 21 a.m.
 Image= $[0, 35]$.

2.



The function is the speed of the plain in mph.
 Domain=The time period from 1 p.m. to 5 p.m.
 Image= $[0, 500]$.

3. The part of the graph corresponding to time after 5 : 25 p.m. can be different depending on the assumptions made.



The function is the speed of the car in mph.
 Domain=The time period from 5 p.m. to 5 : 25 p.m.
 Image= $[0, 55]$.

E. Graphs of tides.

1. The best times are 6 a.m. and 6 p.m.;
the worst times are 12 a.m. and 12 p.m.
2. 6 a.m. and 6 p.m.
3. from 7 a.m. to 11 p.m. and from 7p.m. to 11 a.m.
4. 11 : 30 a.m. and 11 : 30 p.m.

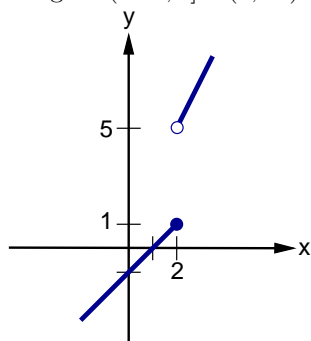
The function is the depth of the harbor in meters.

Domain=The one-day time period.

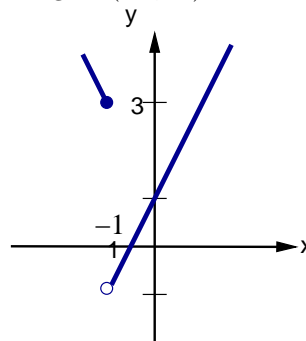
Image=[2, 4].

7. $y = 4x - 2$ is a function of x .
8. $y = 2 + x^2$ is a function of x .
9. $y = \pm\sqrt{x^2 + 1}$ is not a function of x .
10. $y = \pm\sqrt{4x - 1}$ is not a function of x .
11. $y = \sqrt[5]{\frac{100+2x}{3}}$ is a function of x .
12. $y = \sqrt[3]{2+x}$ is a function of x .
13. $y = -3$ is a function of x .
14. $y = \pm\frac{1}{2}\sqrt{1-3x^2}$ is not a function of x .

15. Image = $(-\infty, 1] \cup (5, \infty)$.



16. $f(x) = \begin{cases} 1 - 2x & \text{if } x \leq -1 \\ 2x + 1 & \text{if } x > -1 \end{cases}$
Image = $(-1, \infty)$.



17. $\mathbb{R}, [0, \infty)$.
18. $\mathbb{R}, [1, \infty)$.
19. $f(x) = \frac{1}{\sqrt[3]{x}}$, so $x \neq 0$;
Domain = $\mathbb{R} \setminus \{0\}$, Image = $\mathbb{R} \setminus \{0\}$.
20. $f(x) = \frac{1}{\sqrt[3]{x-1}}$, so $x - 1 \neq 0$;
Domain = $\mathbb{R} \setminus \{1\}$, Image = $\mathbb{R} \setminus \{0\}$.
21. $\mathbb{R}, (0, 3]$.

22. $\mathbb{R}, (0, 4]$.
23. $\mathbb{R} \setminus \{-1\}, \mathbb{R} \setminus \{0\}$.
24. $\mathbb{R} \setminus \{2\}, \mathbb{R}$.
25. $\mathbb{R}, [0, \infty)$.
26. $\mathbb{R}, [1, \infty)$.
27. $[1, \infty), [0, \infty)$.
28. $x^2 + 2x \geq 0, x(x+2) \geq 0$, so Domain = $(-\infty, -2] \cup [0, \infty)$, Image = $[0, \infty)$.
29. \mathbb{R}, \mathbb{R} .
30. \mathbb{R}, \mathbb{R} .
31. $x + 2 > 0$, so Domain = $(-2, \infty)$, Image = $(0, \infty)$.
32. $x - 4 > 0$, so Domain = $(4, \infty)$, Image = $(0, \infty)$.
33. $(f \circ g)(x) = 3 + 2(-2x + 1) = 5 - 4x$, Domain = \mathbb{R} .
34. $(f \circ g)(x) = (x - 4)^2 + 2(x - 4) = x^2 - 8x + 16 + 2x - 8 = x^2 - 6x + 8$,
Domain = \mathbb{R} .
35. $(f \circ g)(x) = |-x| = |x|$, Domain = \mathbb{R} .
36. $(f \circ g)(x) = |x^3 + 2 + 1| = |x^3 + 3|$, Domain = \mathbb{R} .
37. $(g \circ f)(x) = \frac{1}{x+3}$, Domain = $\mathbb{R} \setminus \{-3\}$.
38. $(g \circ f)(x) = \sqrt{\frac{x}{x^2+1}}$, Domain = $[0, \infty)$.
39. $(g \circ f)(x) = (x^4 + 1)^{\frac{1}{3}}$, Domain = \mathbb{R} .
40. $(g \circ f)(x) = (2x + 5)^{\frac{1}{4}}$, Domain = $[-\frac{5}{2}, \infty)$.
41. $f(g(-1)) = f(1) = 2$
42. $(f \circ g)(0) = f(-1) = 3$.
43. $g(f(1)) = g(2) = 5$
44. $(g \circ f)(3) = g(0) = -1$.
45. $f(f(-1)) = f(3) = 0$
46. $(g \circ g)(0) = g(-1) = 1$.
47. $f(x) \geq 0$ for all $x \in \text{Domain}(f)$.
48. $f(x) \geq 2$ for all $x \in \text{Domain}(f)$.
49. The domain of f contains $[0, \infty)$.
50. The domain of f contains $[-2, \infty)$.

51. $f(x) > -1$ for all $x \in \text{Domain}(f)$.
52. $f(x) < 1$ for all $x \in \text{Domain}(f)$.
53. Let $y = x - 1$. Then, $x = y + 1$ so $f(y) = (y + 1)^2$. This implies $f(x + 1) = (x + 1 + 1)^2 = (x + 2)^2$.
54. Let $y = 2x + 1$. Then, $x = \frac{1}{2}(y - 1)$ so $f(y) = \frac{1}{8}(y - 1)^3 - 1$. This implies $f(x + 2) = \frac{1}{8}(x + 1)^3 - 1$.

55. Suppose such functions exist. If we set $x = 0$ we obtain $f(0) + g(y) = 0$ for all $y \in \mathbb{R}$ and if we set $y = 0$ we obtain $f(x) + g(0) = 0$ for all $x \in \mathbb{R}$. Hence,

$$f(x) = -g(0) = -(-f(0)) = f(0) \quad \text{for all } x \in \mathbb{R},$$

while

$$g(y) = -f(0) = -(-g(0)) = g(0) \quad \text{for all } y \in \mathbb{R}.$$

So far we have obtained that f and g are constant functions. On the other hand, this information combined with the identity $f(x) + g(y) = xy$ gives for $y = 1$

$$f(x) = x - g(1) = x - g(0) \quad \text{for all } x \in \mathbb{R},$$

which contradicts the fact that f was found to be constant. Therefore, there do not exist functions f and g satisfying $f(x) + g(y) = xy$ for all $x, y \in \mathbb{R}$.

56. Suppose such functions exist. If we set $x = 0$, we obtain $f(0)g(y) = y$ for all $y \in \mathbb{R}$ and if we set $y = 0$, we obtain $f(x)g(0) = x$ for all $x \in \mathbb{R}$. If we set $x = y = 0$, then we obtain $f(0)g(0) = 0$ and hence either $f(0) = 0$ or $g(0) = 0$. On the other hand $g(0) \neq 0$ since $f(x)g(0) = x$ for all $x \in \mathbb{R}$, and $f(0) \neq 0$ since $f(0)g(y) = y$ for all $y \in \mathbb{R}$. Therefore, there do not exist functions f and g satisfying $f(x)g(y) = x + y$ for all $x, y \in \mathbb{R}$.
57. The condition $f \circ g_1 = h$ implies $2g_1(x) - 3 = x + 6$ for all $x \in \mathbb{R}$. Hence, $g_1(x) = \frac{1}{2}x + \frac{9}{2}$ for all $x \in \mathbb{R}$. From $g_2 \circ f = h$, it follows that $g_2(2x - 3) = x + 6$ for all $x \in \mathbb{R}$, thus $g_2(x) = \frac{x+3}{2} + 6 = \frac{1}{2}x + \frac{15}{2}$ for all $x \in \mathbb{R}$.

Section 2.2: Limits of Functions

1. 0, 1.5, $\lim_{x \rightarrow 2} f(x)$ does not exist.
2. 0.5, 0.5, 0.5.
3. 2, 2, 2.
4. 0, 0, 0.
5. 0, 0, 0, 0.

6. $\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \lim_{x \rightarrow 4} f(x)$ does not exist.

7. $2, +\infty, 2$, does not exist.

8. $\frac{1}{3}, 1, -\infty, \lim_{x \rightarrow 2} f(x)$ does not exist.

9. $0, 0, 0, 0$.

10. $-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}$.

11. $1 - 6 - 4 = -9$.

12. $3(-1)^4 + 5(-1)^3 + 2(-1) - 1 = -5$.

13. $\sqrt{0+3} = \sqrt{3}$.

14. $\sqrt{4^2-3} = \sqrt{13}$.

15. $\frac{2(-1)+1}{-1-3} = \frac{1}{4}$.

16. $\frac{0-6}{3-0+1} = -6$.

17. $4 - 2 \cdot 2 = 0$.

18. $5 - 3^2 = -4$.

19. $\frac{-1}{0+0+1} = -1$.

20. $\frac{0+1}{0-0-1} = -1$.

21. $\sqrt[3]{1-5+1} = \sqrt[3]{-3}$.

22. $\sqrt[3]{2(-2)^3 - (-2)^2 + 3(-2) - 4} = \sqrt[3]{2}$.

23. $\frac{1}{2} - \frac{1}{2} = 0$.

24. $\frac{2}{4+1} = \frac{2}{5}$.

25. $\sqrt[4]{6^2 - 4 \cdot 6 + 4} = 2$.

26. $\sqrt[4]{2^3 + 2 - 1} = \sqrt{3}$.

27. $\frac{(-2)^3 - 2 \cdot (-2) - 3}{-2 + 3} = -7$.

28. $\frac{0-1}{0-1} = 0$.

29. a. $\mathbb{R} \setminus \{1\}$. We can not compute $\lim_{x \rightarrow 1} f(x)$ using **P4** since the denominator of $f(x)$ is zero at $x = 1$.

b. $1, x, \dots, x^n$ are terms of a geometric sequence with $a = 1, r = x$.

$$S = x^n + x^{n-1} + \dots + x + 1$$

$$xS + 1 = x^{n+1} + x^n + \dots + x^2 + x + 1$$

$$xS + 1 = x^{n+1} + S$$

$$S = \frac{x^{n+1} - 1}{x - 1}$$

c. $\lim_{x \rightarrow 1} \frac{x^{n+1} - 1}{x - 1} = \lim_{x \rightarrow 1} (x^n + x^{n-1} + \dots + x + 1) = n + 1$

30. Let $f(x) = x^3 + 2$. Fix an open interval I centered at 3. We want to find an open interval J containing 1, with $J \setminus \{1\}$ contained in the domain of f , such that for every $x \in J \setminus \{1\}$ we have $f(x) \in I$.

Let $2l$ be the length of the interval I , that is, $I = (3-l, 3+l)$. The condition $f(x) \in I$ is equivalent to $|f(x) - 3| < l$, which is in turn equivalent to $3 - l < x^3 + 2 < 3 + l$, $1 - l < x^3 < 1 + l$. This suggests that a good candidate for the interval J is $J = (\sqrt[3]{1-l}, \sqrt[3]{1+l})$. Indeed, $1 \in J$, $J \setminus \{1\}$ is contained in the domain of f , and

$$\begin{aligned} x \in J \setminus \{1\} &\Rightarrow \sqrt[3]{1-l} < x < \sqrt[3]{1+l} \Rightarrow 1-l < x^3 < 1+l \\ &\Rightarrow -l < x^3 - 1 < l \Rightarrow |f(x) - 3| < l \Rightarrow f(x) \in I. \end{aligned}$$

31. Let $f(x) = 3x^3 - 1$. Fix an open interval I centered at 2. We want to find an open interval J containing 1, with $J \setminus \{1\}$ contained in the domain of f , such that for every $x \in J \setminus \{1\}$ we have $f(x) \in I$.

Let $2l$ be the length of the interval I , that is, $I = (2-l, 2+l)$. The condition $f(x) \in I$ is equivalent to $|f(x) - 2| < l$, which is in turn equivalent to $1 - \frac{l}{3} < x^3 < 1 + \frac{l}{3}$. This suggests that a good candidate for the interval J is $J = (\sqrt[3]{1 - \frac{l}{3}}, \sqrt[3]{1 + \frac{l}{3}})$. Indeed, $1 \in J$, $J \setminus \{1\}$ is contained in the domain of f , and

$$\begin{aligned} x \in J \setminus \{1\} &\Rightarrow \sqrt[3]{1 - \frac{l}{3}} < x < \sqrt[3]{1 + \frac{l}{3}} \Rightarrow -\frac{l}{3} < x^3 - 1 < \frac{l}{3} \\ &\Rightarrow -l < 3x^3 - 3 < l \Rightarrow |f(x) - 2| < l \Rightarrow f(x) \in I. \end{aligned}$$

Section 2.3: Continuity

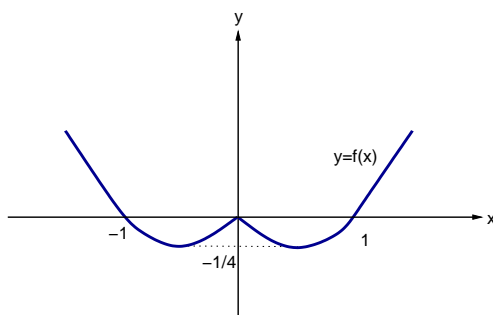
1. Continuous on $\mathbb{R} \setminus \{-1\}$, discontinuous at $x = 1$.
2. Continuous on $\mathbb{R} \setminus \{2\}$, discontinuous at $x = 2$.
3. Continuous on \mathbb{R} .
4. Continuous on \mathbb{R} .
5. $2 \cdot 1^2 - 3 = 3 \cdot 1 - a$, $a = 4$.
6. $a = -1$.
7. $a = 2$.
8. $-2 - a^2 = 1 - 4a$, $a^2 - 4a + 3 = 0$, $a = 1$ or $a = 3$.
9. $f(x) = x^2 - x$.
10. $f(x) = x^3 + x^2$.
11. $f(x) = \begin{cases} 1 & \text{if } x \neq 4 \\ 0 & \text{if } x = 4 \end{cases}$
12. $f(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$
13. $f(x) = \begin{cases} 0 & \text{if } x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$
14. $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ x + 1 & \text{if } x > 2 \end{cases}$
15. Solving for $f(x)$ we obtain that for each $x \in \mathbb{R}$, $f(x)$ is either x or $-x$. Hence the graph of f is contained in the lines $y = x$ and $y = -x$. Since f is continuous, we have four possibilities: $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = |x|$, and $f_4(x) = -|x|$.

Chapter 2 Review

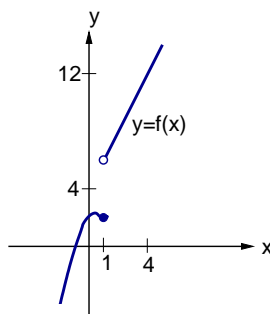
1. $y = \pm\sqrt{x^3 - 2}$ is not a function of x .
2. $y = \sqrt[3]{2 + x^2}$ is a function of x .
3. $y = \frac{5}{2}x + \frac{1}{2}$ is a function of x .

4. $y = -\frac{2}{x}$ is a function of x .

5. $f(x) = \begin{cases} x^2 + x & \text{if } x \leq 0 \\ x^2 - x & \text{if } x > 0 \end{cases}$, $\text{Image}(f) = [-\frac{1}{4}, \infty)$.



6. $f(x) = \begin{cases} 2 + x - x^2 & \text{if } x \leq 1 \\ 2x + 4 & \text{if } x > 1 \end{cases}$, $\text{Image}(f) = (-\infty, \frac{9}{4}] \cup (6, \infty)$.



7. $\text{Domain}(f) = \mathbb{R}$, $\text{Image}(f) = [\frac{7}{4}, \infty)$.

8. $\text{Domain}(f) = \mathbb{R}$, $\text{Image}(f) = (-\infty, 4]$.

9. $\text{Domain}(f) = [0, \infty)$, $\text{Image}(f) = [0, \infty)$.

10. $x^2 - 2x - 3 = (x + 1)(x - 3) \geq 0$.
 $\text{Domain}(f) = (-\infty, -1] \cup [3, \infty)$, $\text{Image}(f) = [0, \infty)$.

11. $\text{Domain}(f) = \mathbb{R}$, $\text{Image}(f) = \mathbb{R}$.

12. $\text{Domain}(f) = \mathbb{R}$, $\text{Image}(f) = \mathbb{R}$.

13. $\text{Domain}(f) = \mathbb{R} \setminus \{-2\}$, $\text{Image}(f) = \{-1, 1\}$.

14. $x - 1 \geq 0$, so $\text{Domain}(f) = [1, \infty)$, $\text{Image}(f) = \{1\}$.

15. $\text{Domain}(f) = \mathbb{R} \setminus \{-3, 3\}$, $\text{Image}(f) = \mathbb{R} \setminus \{0, -\frac{1}{6}\}$.

16. $x - 2 \neq 0$, so $\text{Domain}(f) = \mathbb{R} \setminus \{2\}$, $\text{Image}(f) = \mathbb{R}$.

17. $f(g(x)) = \frac{x^2+2x}{x^2+2x-3}$, Domain $(f \circ g) = \mathbb{R} \setminus \{-3, 1\}$.
18. $f(g(x)) = \frac{\sqrt{x^2-4x}}{x^2-4x-1}$, $x^2 - 4x = x(x - 4) \geq 0$ and $x^2 - 4x - 1 = (x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) \neq 0$, so Domain $(f \circ g) = (-\infty, 2 - \sqrt{5}) \cup (2 - \sqrt{5}, 0] \cup [4, 2 + \sqrt{5}) \cup (2 + \sqrt{5}, \infty)$.
19. $g(f(x)) = \sqrt{x - 3}$, Domain $(g \circ f) = [3, \infty)$.
20. $g(f(x)) = \sqrt{x^2 + x}$, $x^2 + x = x(x + 1) \geq 0$, so Domain $(g \circ f) = (-\infty, -1] \cup [0, \infty)$.
21. $f(g(x)) = \begin{cases} 2x + 5 & \text{if } x < -1 \\ x^2 + 2x + 4 & \text{if } x \geq -1 \end{cases}$, $g(f(x)) = \begin{cases} 2x + 4 & \text{if } x < 0 \\ x^2 + 4 & \text{if } x \geq 0 \end{cases}$.
22. $f(g(x)) = \begin{cases} -8 & \text{if } x \leq -2 \\ 3x - 2 & \text{if } -2 < x \leq 0, \\ -2 & \text{if } x > 0 \end{cases}$, $g(f(x)) = -3$.
23. f is not defined at $-2, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}$.
24. $-\frac{1}{2}, -\frac{1}{2}, 1, \lim_{x \rightarrow \frac{1}{2}} f(x)$ does not exist.
25. $1, -\infty, 1$, does not exist.
26. $0, -1, \infty, \lim_{x \rightarrow 3} f(x)$ does not exist.
27. $0, 0, 0, 0$.
28. $2, 1, 1, 1$.
29. 56 .
30. $100 - 3(-1)^2 + (-1)^3 - 7(-1)^5 = 101$.
31. -1
32. $\sqrt{3 \cdot 1^5 - 2 \cdot 1 + 2} = \sqrt{3}$.
33. 0
34. $\frac{3+2}{3^2-6} = \frac{5}{3}$.
35. -3 .
36. $\frac{3 \cdot 0 - 2}{0^3 - 1} + \sqrt[3]{2 - 0^2 - 0} = 2 + \sqrt[3]{2}$.
37. $f(x) = \begin{cases} x + 1 & \text{if } x \leq 0 \\ -x & \text{if } x > 0 \end{cases}$

$$38. f(x) = \begin{cases} x & \text{if } x < -1 \\ x + 1 & \text{if } x \geq -1 \end{cases}.$$

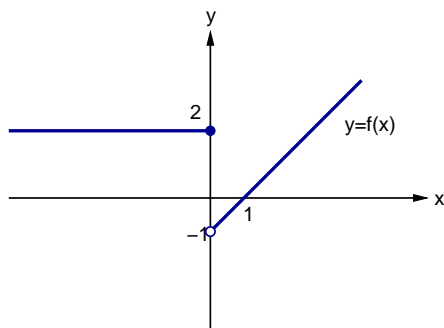
$$39. f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}.$$

$$40. f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}.$$

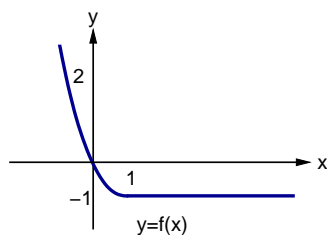
$$41. f(x) = x^2 + 3x - 1.$$

$$42. f(x) = \frac{x}{x^2+1}.$$

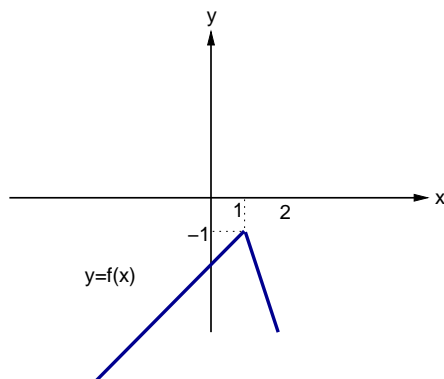
43. f is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0.



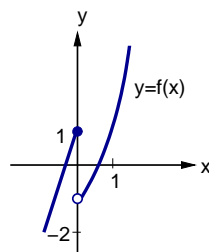
44. f is continuous on \mathbb{R} .



45. f is continuous on \mathbb{R} .



46. f is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at $\{0\}$.



47. $a = -1$.

48. $a^2 + 2 = 1 - 1 + 6$, $a^2 = 4$, $a = \pm 2$.

49. If $f(x) = ax + b$, $(f \circ f)(x) = x$ for all $x \in \mathbb{R}$ is equivalent to $a(ax + b) + b = x$ for all $x \in \mathbb{R}$. Hence, $(a^2 - 1)x + ab + b = 0$ for all $x \in \mathbb{R}$. The latter holds if and only if $a^2 - 1 = 0$ and $ab + b = 0$. The solutions for this system in a and b are $a = 1$, $b = 0$ or $a = -1$, b arbitrary real number. Consequently, $f(x) = x$ or $f(x) = -x + b$.