

1.1 Concepts Review

1. $L; c$
2. 6
3. L ; right
4. $\lim_{x \rightarrow c} f(x) = M$

Problem Set 1.1

1. $\lim_{x \rightarrow 3} (x - 5) = -2$
2. $\lim_{t \rightarrow -1} (1 - 2t) = 3$
3. $\lim_{x \rightarrow -2} (x^2 + 2x - 1) = (-2)^2 + 2(-2) - 1 = -1$
4. $\lim_{x \rightarrow -2} (x^2 + 2t - 1) = (-2)^2 + 2t - 1 = 3 + 2t$
5. $\lim_{t \rightarrow -1} (t^2 - 1) = ((-1)^2 - 1) = 0$
6. $\lim_{t \rightarrow -1} (t^2 - x^2) = ((-1)^2 - x^2) = 1 - x^2$
7. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2}$
 $= \lim_{x \rightarrow 2} (x + 2)$
 $= 2 + 2 = 4$
8. $\lim_{t \rightarrow -7} \frac{t^2 + 4t - 21}{t + 7}$
 $= \lim_{t \rightarrow -7} \frac{(t + 7)(t - 3)}{t + 7}$
 $= \lim_{t \rightarrow -7} (t - 3)$
 $= -7 - 3 = -10$
9. $\lim_{x \rightarrow -1} \frac{x^3 - 4x^2 + x + 6}{x + 1}$
 $= \lim_{x \rightarrow -1} \frac{(x + 1)(x^2 - 5x + 6)}{x + 1}$
 $= \lim_{x \rightarrow -1} (x^2 - 5x + 6)$
 $= (-1)^2 - 5(-1) + 6$
 $= 12$
10. $\lim_{x \rightarrow 0} \frac{x^4 + 2x^3 - x^2}{x^2}$
 $= \lim_{x \rightarrow 0} (x^2 + 2x - 1) = -1$
11. $\lim_{x \rightarrow -t} \frac{x^2 - t^2}{x + t} = \lim_{x \rightarrow -t} \frac{(x + t)(x - t)}{x + t}$
 $= \lim_{x \rightarrow -t} (x - t)$
 $= -t - t = -2t$
12. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$
 $= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3}$
 $= \lim_{x \rightarrow 3} (x + 3)$
 $= 3 + 3 = 6$
13. $\lim_{t \rightarrow 2} \frac{\sqrt{(t + 4)(t - 2)^4}}{(3t - 6)^2}$
 $= \lim_{t \rightarrow 2} \frac{(t - 2)^2 \sqrt{t + 4}}{9(t - 2)^2}$
 $= \lim_{t \rightarrow 2} \frac{\sqrt{t + 4}}{9}$
 $= \frac{\sqrt{2 + 4}}{9} = \frac{\sqrt{6}}{9}$
14. $\lim_{t \rightarrow 7^+} \frac{\sqrt{(t - 7)^3}}{t - 7}$
 $= \lim_{t \rightarrow 7^+} \frac{(t - 7)\sqrt{t - 7}}{t - 7}$
 $= \lim_{t \rightarrow 7^+} \sqrt{t - 7}$
 $= \sqrt{7 - 7} = 0$

$$15. \lim_{x \rightarrow 3} \frac{x^4 - 18x^2 + 81}{(x-3)^2} = \lim_{x \rightarrow 3} \frac{(x^2 - 9)^2}{(x-3)^2}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)^2(x+3)^2}{(x-3)^2} = \lim_{x \rightarrow 3} (x+3)^2 = (3+3)^2$$

$$= 36$$

$$16. \lim_{u \rightarrow 1} \frac{(3u+4)(2u-2)^3}{(u-1)^2} = \lim_{u \rightarrow 1} \frac{8(3u+4)(u-1)^3}{(u-1)^2}$$

$$= \lim_{u \rightarrow 1} 8(3u+4)(u-1) = 8[3(1)+4](1-1) = 0$$

$$17. \lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{4+4h+h^2-4}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2+4h}{h} = \lim_{h \rightarrow 0} (h+4) = 4$$

$$18. \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 2xh}{h} = \lim_{h \rightarrow 0} (h + 2x) = 2x$$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.49999992
-1.	0.420735
-0.1	0.499167
-0.01	0.499992
-0.001	0.49999992

$$\lim_{x \rightarrow 0} \frac{\sin x}{2x} = 0.5$$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.0002499998
-1.	-0.229849
-0.1	-0.0249792
-0.01	-0.00249998
-0.001	-0.0002499998

$$\lim_{t \rightarrow 0} \frac{1 - \cos t}{2t} = 0$$

21.

x	$(x - \sin x)^2 / x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.77775×10^{-10}
0.001	2.77778×10^{-14}
-1.	0.0251314
-0.1	2.775×10^{-6}
-0.01	2.77775×10^{-10}
-0.001	2.77778×10^{-14}

$$\lim_{x \rightarrow 0} \frac{(x - \sin x)^2}{x^2} = 0$$

22.

x	$(1 - \cos x)^2 / x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
-1.	0.211322
-0.1	0.00249584
-0.01	0.0000249996
-0.001	2.5×10^{-7}

$$\lim_{x \rightarrow 0} \frac{(1 - \cos x)^2}{x^2} = 0$$

23.

t	$(t^2 - 1) / (\sin(t - 1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$$\lim_{t \rightarrow 1} \frac{t^2 - 1}{\sin(t - 1)} = 2$$

24.

x	$\frac{x - \sin(x-3) - 3}{x-3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x \rightarrow 3} \frac{x - \sin(x-3) - 3}{x-3} = 0$$

25.

x	$(1 + \sin(x - 3\pi/2)) / (x - \pi)$
$1. + \pi$	0.4597
$0.1 + \pi$	0.0500
$0.01 + \pi$	0.0050
$0.001 + \pi$	0.0005
$-1. + \pi$	-0.4597
$-0.1 + \pi$	-0.0500
$-0.01 + \pi$	-0.0050
$-0.001 + \pi$	-0.0005

$$\lim_{x \rightarrow \pi} \frac{1 + \sin\left(x - \frac{3\pi}{2}\right)}{x - \pi} = 0$$

26.

t	$(1 - \cot t) / (1/t)$
1.	0.357907
0.1	-0.896664
0.01	-0.989967
0.001	-0.999
-1.	-1.64209
-0.1	-1.09666
-0.01	-1.00997
-0.001	-1.001

$$\lim_{t \rightarrow 0} \frac{1 - \cot t}{\frac{1}{t}} = -1$$

27.

x	$(x - \pi/4)^2 / (\tan x - 1)^2$
$1. + \frac{\pi}{4}$	0.0320244
$0.1 + \frac{\pi}{4}$	0.201002
$0.01 + \frac{\pi}{4}$	0.245009
$0.001 + \frac{\pi}{4}$	0.2495
$-1. + \frac{\pi}{4}$	0.674117
$-0.1 + \frac{\pi}{4}$	0.300668
$-0.01 + \frac{\pi}{4}$	0.255008
$-0.001 + \frac{\pi}{4}$	0.2505

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\left(x - \frac{\pi}{4}\right)^2}{(\tan x - 1)^2} = 0.25$$

28.

u	$(2 - 2 \sin u) / 3u$
$1. + \frac{\pi}{2}$	0.11921
$0.1 + \frac{\pi}{2}$	0.00199339
$0.01 + \frac{\pi}{2}$	0.0000210862
$0.001 + \frac{\pi}{2}$	2.12072×10^{-7}
$-1. + \frac{\pi}{2}$	0.536908
$-0.1 + \frac{\pi}{2}$	0.00226446
$-0.01 + \frac{\pi}{2}$	0.0000213564
$-0.001 + \frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u \rightarrow \frac{\pi}{2}} \frac{2 - 2 \sin u}{3u} = 0$$

29. a. $\lim_{x \rightarrow -3} f(x) = 2$

b. $f(-3) = 1$

c. $f(-1)$ does not exist.

d. $\lim_{x \rightarrow -1} f(x) = \frac{5}{2}$

e. $f(1) = 2$

f. $\lim_{x \rightarrow 1} f(x)$ does not exist.

g. $\lim_{x \rightarrow 1^-} f(x) = 2$

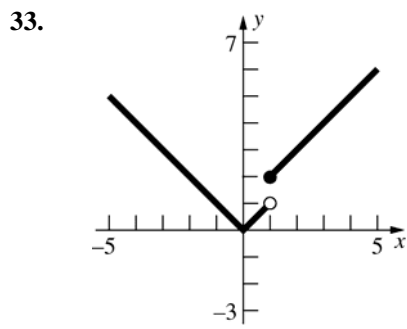
h. $\lim_{x \rightarrow 1^+} f(x) = 1$

i. $\lim_{x \rightarrow -1^+} f(x) = \frac{5}{2}$

30. a. $\lim_{x \rightarrow -3} f(x)$ does not exist.
 b. $f(-3) = 1$
 c. $f(-1) = 1$
 d. $\lim_{x \rightarrow -1} f(x) = 2$
 e. $f(1) = 1$
 f. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 g. $\lim_{x \rightarrow 1^-} f(x) = 1$
 h. $\lim_{x \rightarrow 1^+} f(x)$ does not exist.
 i. $\lim_{x \rightarrow -1^+} f(x) = 2$

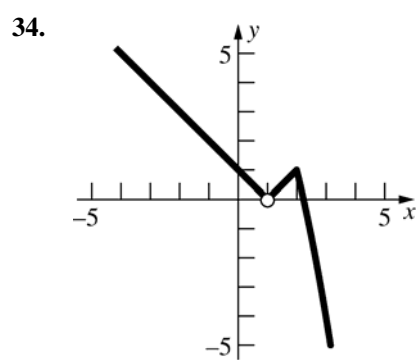
31. a. $f(-3) = 2$
 b. $f(3)$ is undefined.
 c. $\lim_{x \rightarrow -3^-} f(x) = 2$
 d. $\lim_{x \rightarrow -3^+} f(x) = 4$
 e. $\lim_{x \rightarrow -3} f(x)$ does not exist.
 f. $\lim_{x \rightarrow 3^+} f(x)$ does not exist.

32. a. $\lim_{x \rightarrow -1^-} f(x) = -2$
 b. $\lim_{x \rightarrow -1^+} f(x) = -2$
 c. $\lim_{x \rightarrow -1} f(x) = -2$
 d. $f(-1) = -2$
 e. $\lim_{x \rightarrow 1} f(x) = 0$
 f. $f(1) = 0$



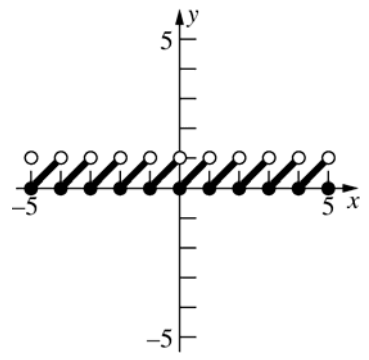
- a. $\lim_{x \rightarrow 0} f(x) = 0$

- b. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 c. $f(1) = 2$
 d. $\lim_{x \rightarrow 1^+} f(x) = 2$



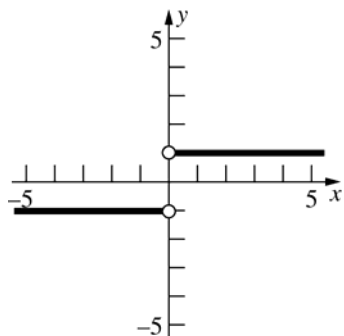
- a. $\lim_{x \rightarrow 1} g(x) = 0$
 b. $g(1)$ does not exist.
 c. $\lim_{x \rightarrow 2} g(x) = 1$
 d. $\lim_{x \rightarrow 2^+} g(x) = 1$

35. $f(x) = x - \lceil x \rceil$



- a. $f(0) = 0$
 b. $\lim_{x \rightarrow 0} f(x)$ does not exist.
 c. $\lim_{x \rightarrow 0^-} f(x) = 1$
 d. $\lim_{x \rightarrow \frac{1}{2}} f(x) = \frac{1}{2}$

36. $f(x) = \frac{x}{|x|}$



a. $f(0)$ does not exist.

b. $\lim_{x \rightarrow 0} f(x)$ does not exist.

c. $\lim_{x \rightarrow 0^-} f(x) = -1$

d. $\lim_{x \rightarrow \frac{1}{2}} f(x) = 1$

37. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|}$ does not exist.

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = -2 \text{ and } \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = 2$$

38. $\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})}$$

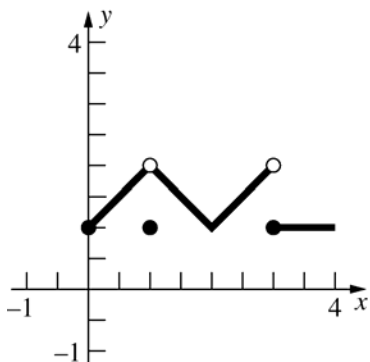
$$= \lim_{x \rightarrow 0} \frac{x + 2 - 2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+2} + \sqrt{2})}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{0+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

39. a. $\lim_{x \rightarrow 1} f(x)$ does not exist.

b. $\lim_{x \rightarrow 0} f(x) = 0$

40.



41. $\lim_{x \rightarrow a} f(x)$ exists for $a = -1, 0, 1$.

42. The changed values will not change $\lim_{x \rightarrow a} f(x)$ at any a . As x approaches a , the limit is still a^2 .

43. a. $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$ does not exist.
 $\lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} = -1$ and $\lim_{x \rightarrow 1^+} \frac{|x-1|}{x-1} = 1$

b. $\lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} = -1$

c. $\lim_{x \rightarrow 1^-} \frac{x^2 - |x-1| - 1}{|x-1|} = -3$

d. $\lim_{x \rightarrow 1^-} \left[\frac{1}{x-1} - \frac{1}{|x-1|} \right]$ does not exist.

44. a. $\lim_{x \rightarrow 1^+} \sqrt{x - \lfloor x \rfloor} = 0$

b. $\lim_{x \rightarrow 0^+} \left[\frac{1}{x} \right]$ does not exist.

c. $\lim_{x \rightarrow 0^+} x(-1)^{\lfloor 1/x \rfloor} = 0$

d. $\lim_{x \rightarrow 0^+} \lfloor x \rfloor (-1)^{\lfloor 1/x \rfloor} = 0$

45. a) 1 b) 0

c) -1 d) -1

46. a) Does not exist b) 0

c) 1 d) 0.556

47. $\lim_{x \rightarrow 0} \sqrt{x}$ does not exist since \sqrt{x} is not defined for $x < 0$.

48. $\lim_{x \rightarrow 0^+} x^x = 1$

49. $\lim_{x \rightarrow 0} \sqrt{|x|} = 0$

50. $\lim_{x \rightarrow 0} |x|^x = 1$

51. $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x} = \frac{1}{2}$

$$52. \lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \frac{5}{3}$$

$$53. \lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right) \text{ does not exist.}$$

$$54. \lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) = 0$$

$$55. \lim_{x \rightarrow 1} \frac{x^3 - 1}{\sqrt{2x + 2} - 2} = 6$$

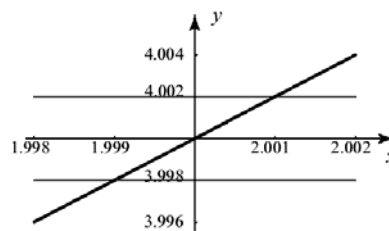
$$56. \lim_{x \rightarrow 0} \frac{x \sin 2x}{\sin(x^2)} = 2$$

$$57. \lim_{x \rightarrow 2^-} \frac{x^2 - x - 2}{|x - 2|} = -3$$

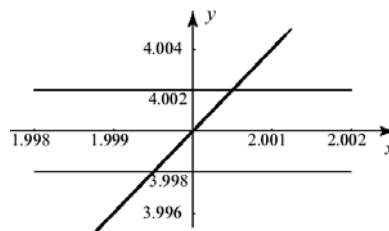
$$58. \lim_{x \rightarrow 1^+} \frac{2}{1 + 2^{1/(x-1)}} = 0$$

$$59. \lim_{x \rightarrow 0} \sqrt{x}; \text{ The computer gives a value of 0, but } \lim_{x \rightarrow 0^-} \sqrt{x} \text{ does not exist.}$$

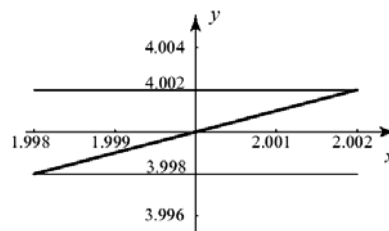
7. If x is within 0.001 of 2, then $2x$ is within 0.002 of 4.



8. If x is within 0.0005 of 2, then x^2 is within 0.002 of 4.



9. If x is within 0.0019 of 2, then $\sqrt{8x}$ is within 0.002 of 4.



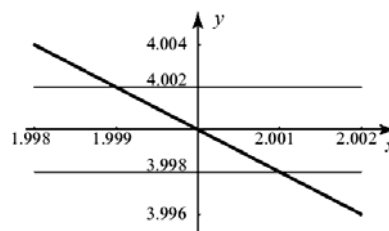
1.2 Concepts Review

- $L - \varepsilon; L + \varepsilon$
- $0 < |x - a| < \delta; |f(x) - L| < \varepsilon$
- $\frac{\varepsilon}{3}$
- $ma + b$

Problem Set 1.2

- $0 < |t - a| < \delta \Rightarrow |f(t) - M| < \varepsilon$
- $0 < |u - b| < \delta \Rightarrow |g(u) - L| < \varepsilon$
- $0 < |z - d| < \delta \Rightarrow |h(z) - P| < \varepsilon$
- $0 < |y - e| < \delta \Rightarrow |\phi(y) - B| < \varepsilon$
- $0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$
- $0 < t - a < \delta \Rightarrow |g(t) - D| < \varepsilon$

10. If x is within 0.001 of 2, then $\frac{8}{x}$ is within 0.002 of 4.



11. $0 < |x - 0| < \delta \Rightarrow |(2x - 1) - (-1)| < \varepsilon$
 $|2x - 1 + 1| < \varepsilon \Leftrightarrow |2x| < \varepsilon$
 $\Leftrightarrow 2|x| < \varepsilon$
 $\Leftrightarrow |x| < \frac{\varepsilon}{2}$
 $\delta = \frac{\varepsilon}{2}; 0 < |x - 0| < \delta$
 $|(2x - 1) - (-1)| = |2x| = 2|x| < 2\delta = \varepsilon$

$$\begin{aligned}
12. \quad 0 < |x+21| < \delta &\Rightarrow |(3x-1) - (-64)| < \varepsilon \\
|3x-1+64| < \varepsilon &\Leftrightarrow |3x+63| < \varepsilon \\
&\Leftrightarrow |3(x+21)| < \varepsilon \\
&\Leftrightarrow 3|x+21| < \varepsilon \\
&\Leftrightarrow |x+21| < \frac{\varepsilon}{3}
\end{aligned}$$

$$\delta = \frac{\varepsilon}{3}; 0 < |x+21| < \delta$$

$$|(3x-1) - (-64)| = |3x+63| = 3|x+21| < 3\delta = \varepsilon$$

$$\begin{aligned}
13. \quad 0 < |x-5| < \delta &\Rightarrow \left| \frac{x^2-25}{x-5} - 10 \right| < \varepsilon \\
\left| \frac{x^2-25}{x-5} - 10 \right| < \varepsilon &\Leftrightarrow \left| \frac{(x-5)(x+5)}{x-5} - 10 \right| < \varepsilon \\
&\Leftrightarrow |x+5-10| < \varepsilon \\
&\Leftrightarrow |x-5| < \varepsilon
\end{aligned}$$

$$\delta = \varepsilon; 0 < |x-5| < \delta$$

$$\begin{aligned}
\left| \frac{x^2-25}{x-5} - 10 \right| &= \left| \frac{(x-5)(x+5)}{x-5} - 10 \right| = |x+5-10| \\
&= |x-5| < \delta = \varepsilon
\end{aligned}$$

$$\begin{aligned}
14. \quad 0 < |x-0| < \delta &\Rightarrow \left| \frac{2x^2-x}{x} - (-1) \right| < \varepsilon \\
\left| \frac{2x^2-x}{x} + 1 \right| < \varepsilon &\Leftrightarrow \left| \frac{x(2x-1)}{x} + 1 \right| < \varepsilon \\
&\Leftrightarrow |2x-1+1| < \varepsilon \\
&\Leftrightarrow |2x| < \varepsilon \\
&\Leftrightarrow 2|x| < \varepsilon \\
&\Leftrightarrow |x| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\delta = \frac{\varepsilon}{2}; 0 < |x-0| < \delta$$

$$\begin{aligned}
\left| \frac{2x^2-x}{x} - (-1) \right| &= \left| \frac{x(2x-1)}{x} + 1 \right| = |2x-1+1| \\
&= |2x| = 2|x| < 2\delta = \varepsilon
\end{aligned}$$

$$\begin{aligned}
15. \quad 0 < |x-5| < \delta &\Rightarrow \left| \frac{2x^2-11x+5}{x-5} - 9 \right| < \varepsilon \\
\left| \frac{2x^2-11x+5}{x-5} - 9 \right| < \varepsilon &\Leftrightarrow \left| \frac{(2x-1)(x-5)}{x-5} - 9 \right| < \varepsilon \\
&\Leftrightarrow |2x-1-9| < \varepsilon \\
&\Leftrightarrow |2(x-5)| < \varepsilon \\
&\Leftrightarrow |x-5| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\delta = \frac{\varepsilon}{2}; 0 < |x-5| < \delta$$

$$\begin{aligned}
\left| \frac{2x^2-11x+5}{x-5} - 9 \right| &= \left| \frac{(2x-1)(x-5)}{x-5} - 9 \right| \\
&= |2x-1-9| = |2(x-5)| = 2|x-5| < 2\delta = \varepsilon
\end{aligned}$$

$$\begin{aligned}
16. \quad 0 < |x-1| < \delta &\Rightarrow |\sqrt{2x} - \sqrt{2}| < \varepsilon \\
|\sqrt{2x} - \sqrt{2}| < \varepsilon &\Leftrightarrow \left| \frac{(\sqrt{2x} - \sqrt{2})(\sqrt{2x} + \sqrt{2})}{\sqrt{2x} + \sqrt{2}} \right| < \varepsilon \\
&\Leftrightarrow \left| \frac{2x-2}{\sqrt{2x} + \sqrt{2}} \right| < \varepsilon \\
&\Leftrightarrow 2 \left| \frac{x-1}{\sqrt{2x} + \sqrt{2}} \right| < \varepsilon
\end{aligned}$$

$$\delta = \frac{\sqrt{2}\varepsilon}{2}; 0 < |x-1| < \delta$$

$$\begin{aligned}
|\sqrt{2x} - \sqrt{2}| &= \left| \frac{(\sqrt{2x} - \sqrt{2})(\sqrt{2x} + \sqrt{2})}{\sqrt{2x} + \sqrt{2}} \right| \\
&= \left| \frac{2x-2}{\sqrt{2x} + \sqrt{2}} \right| \\
\frac{2|x-1|}{\sqrt{2x} + \sqrt{2}} &\leq \frac{2|x-1|}{\sqrt{2}} < \frac{2\delta}{\sqrt{2}} = \varepsilon
\end{aligned}$$

$$\begin{aligned}
17. \quad 0 < |x-4| < \delta &\Rightarrow \left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| < \varepsilon \\
\left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| < \varepsilon &\Leftrightarrow \left| \frac{\sqrt{2x-1} - \sqrt{7(x-3)}}{\sqrt{x-3}} \right| < \varepsilon \\
&\Leftrightarrow \left| \frac{(\sqrt{2x-1} - \sqrt{7(x-3)})(\sqrt{2x-1} + \sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1} + \sqrt{7(x-3)})} \right| < \varepsilon \\
&\Leftrightarrow \left| \frac{2x-1 - (7x-21)}{\sqrt{x-3}(\sqrt{2x-1} + \sqrt{7(x-3)})} \right| < \varepsilon \\
&\Leftrightarrow \left| \frac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1} + \sqrt{7(x-3)})} \right| < \varepsilon
\end{aligned}$$

$$\Leftrightarrow |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$$

To bound $\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree that

$$\delta \leq \frac{1}{2}. \text{ If } \delta \leq \frac{1}{2}, \text{ then } \frac{7}{2} < x < \frac{9}{2}, \text{ so}$$

$$0.65 < \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < 1.65 \text{ and}$$

$$\text{hence } |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$$

$$\Leftrightarrow |x-4| < \frac{\varepsilon}{1.65}$$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{1.65}\right\}, \quad 0 < |x-4| < \delta$$

$$\left| \frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7} \right| = |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$$

$$< |x-4|(1.65) < 1.65\delta \leq \varepsilon$$

since $\delta = \frac{1}{2}$ only when $\frac{1}{2} \leq \frac{\varepsilon}{1.65}$ so $1.65\delta \leq \varepsilon$.

$$18. \quad 0 < |x-1| < \delta \Rightarrow \left| \frac{14x^2 - 20x + 6}{x-1} - 8 \right| < \varepsilon$$

$$\left| \frac{14x^2 - 20x + 6}{x-1} - 8 \right| < \varepsilon \Leftrightarrow \left| \frac{2(7x-3)(x-1)}{x-1} - 8 \right| < \varepsilon$$

$$\Leftrightarrow |2(7x-3) - 8| < \varepsilon$$

$$\Leftrightarrow |14(x-1)| < \varepsilon$$

$$\Leftrightarrow 14|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{14}$$

$$\delta = \frac{\varepsilon}{14}; \quad 0 < |x-1| < \delta$$

$$\left| \frac{14x^2 - 20x + 6}{x-1} - 8 \right| = \left| \frac{2(7x-3)(x-1)}{x-1} - 8 \right|$$

$$= |2(7x-3) - 8|$$

$$= |14(x-1)| = 14|x-1| < 14\delta = \varepsilon$$

$$19. \quad 0 < |x-1| < \delta \Rightarrow \left| \frac{10x^3 - 26x^2 + 22x - 6}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\left| \frac{10x^3 - 26x^2 + 22x - 6}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{(10x-6)(x-1)^2}{(x-1)^2} - 4 \right| < \varepsilon$$

$$\Leftrightarrow |10x - 6 - 4| < \varepsilon$$

$$\Leftrightarrow |10(x-1)| < \varepsilon$$

$$\Leftrightarrow 10|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{10}$$

$$\delta = \frac{\varepsilon}{10}; \quad 0 < |x-1| < \delta$$

$$\left| \frac{10x^3 - 26x^2 + 22x - 6}{(x-1)^2} - 4 \right| = \left| \frac{(10x-6)(x-1)^2}{(x-1)^2} - 4 \right|$$

$$= |10x - 6 - 4| = |10(x-1)|$$

$$= 10|x-1| < 10\delta = \varepsilon$$

$$20. \quad 0 < |x-1| < \delta \Rightarrow |(2x^2 + 1) - 3| < \varepsilon$$

$$|2x^2 + 1 - 3| = |2x^2 - 2| = 2|x+1||x-1|$$

To bound $|2x+2|$, agree that $\delta \leq 1$.

$$|x-1| < \delta \text{ implies}$$

$$|2x+2| = |2x-2+4|$$

$$\leq |2x-2| + |4|$$

$$< 2 + 4 = 6$$

$$\delta \leq \frac{\varepsilon}{6}; \quad \delta = \min\left\{1, \frac{\varepsilon}{6}\right\}; \quad 0 < |x-1| < \delta$$

$$|(2x^2 + 1) - 3| = |2x^2 - 2|$$

$$= |2x+2||x-1| < 6 \cdot \left(\frac{\varepsilon}{6}\right) = \varepsilon$$

$$21. \quad 0 < |x+1| < \delta \Rightarrow |(x^2 - 2x - 1) - 2| < \varepsilon$$

$$|x^2 - 2x - 1 - 2| = |x^2 - 2x - 3| = |x+1||x-3|$$

To bound $|x-3|$, agree that $\delta \leq 1$.

$|x+1| < \delta$ implies

$$|x-3| = |x+1-4| \leq |x+1| + |-4| < 1 + 4 = 5$$

$$\delta \leq \frac{\varepsilon}{5}; \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}; 0 < |x+1| < \delta$$

$$|(x^2 - 2x - 1) - 2| = |x^2 - 2x - 3|$$

$$= |x+1||x-3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

$$22. \quad 0 < |x| < \delta \Rightarrow |x^4 - 0| = |x^4| < \varepsilon$$

$|x^4| = |x||x^3|$. To bound $|x^3|$, agree that

$\delta \leq 1$. $|x| < \delta \leq 1$ implies $|x^3| = |x|^3 \leq 1$ so

$\delta \leq \varepsilon$.

$$\delta = \min\{1, \varepsilon\}; 0 < |x| < \delta \Rightarrow |x^4| = |x||x^3| < \varepsilon \cdot 1$$

$= \varepsilon$

23. Choose $\varepsilon > 0$. Then since $\lim_{x \rightarrow c} f(x) = L$, there is

some $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon.$$

Since $\lim_{x \rightarrow c} f(x) = M$, there is some $\delta_2 > 0$ such

$$\text{that } 0 < |x - c| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and choose x_0 such that

$$0 < |x_0 - c| < \delta.$$

$$\text{Thus, } |f(x_0) - L| < \varepsilon \Rightarrow -\varepsilon < f(x_0) - L < \varepsilon$$

$$\Rightarrow -f(x_0) - \varepsilon < -L < -f(x_0) + \varepsilon$$

$$\Rightarrow f(x_0) - \varepsilon < L < f(x_0) + \varepsilon.$$

Similarly,

$$f(x_0) - \varepsilon < M < f(x_0) + \varepsilon.$$

Thus,

$$-2\varepsilon < L - M < 2\varepsilon. \text{ As } \varepsilon \Rightarrow 0, L - M \Rightarrow 0, \text{ so } L = M.$$

24. Since $\lim_{x \rightarrow c} G(x) = 0$, then given any $\varepsilon > 0$, we

can find $\delta > 0$ such that whenever

$$|x - c| < \delta, |G(x)| < \varepsilon.$$

Take any $\varepsilon > 0$ and the corresponding δ that

works for $G(x)$, then $|x - c| < \delta$ implies

$$|F(x) - 0| = |F(x)| \leq |G(x)| < \varepsilon \text{ since}$$

$$\lim_{x \rightarrow c} G(x) = 0.$$

$x \rightarrow c$

$$\text{Thus, } \lim_{x \rightarrow c} F(x) = 0.$$

25. For all $x \neq 0$, $0 \leq \sin^2\left(\frac{1}{x}\right) \leq 1$ so

$$x^4 \sin^2\left(\frac{1}{x}\right) \leq x^4 \text{ for all } x \neq 0. \text{ By Problem 18,}$$

$$\lim_{x \rightarrow 0} x^4 = 0, \text{ so, by Problem 20,}$$

$$\lim_{x \rightarrow 0} x^4 \sin^2\left(\frac{1}{x}\right) = 0.$$

$$26. \quad 0 < x < \delta \Rightarrow |\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x} < \varepsilon$$

For $x > 0$, $(\sqrt{x})^2 = x$.

$$\sqrt{x} < \varepsilon \Leftrightarrow (\sqrt{x})^2 = x < \varepsilon^2$$

$$\delta = \varepsilon^2; 0 < x < \delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

$$27. \quad \lim_{x \rightarrow 0^+} |x|: 0 < x < \delta \Rightarrow |x| - 0 < \varepsilon$$

For $x \geq 0$, $|x| = x$.

$$\delta = \varepsilon; 0 < x < \delta \Rightarrow |x| - 0 = |x| = x < \delta = \varepsilon$$

Thus, $\lim_{x \rightarrow 0^+} |x| = 0$.

$$\lim_{x \rightarrow 0^-} |x|: 0 < 0 - x < \delta \Rightarrow |x| - 0 < \varepsilon$$

For $x < 0$, $|x| = -x$; note also that $||x|| = |x|$ since $|x| \geq 0$.

$$\delta = \varepsilon; 0 < -x < \delta \Rightarrow ||x|| = |x| = -x < \delta = \varepsilon$$

Thus, $\lim_{x \rightarrow 0^-} |x| = 0$,

$$\text{since } \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^-} |x| = 0, \lim_{x \rightarrow 0} |x| = 0.$$

28. Choose $\varepsilon > 0$. Since $\lim_{x \rightarrow a} g(x) = 0$ there is some

$\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |g(x) - 0| < \frac{\varepsilon}{B}.$$

Let $\delta = \min\{1, \delta_1\}$, then $|f(x)| < B$ for

$|x - a| < \delta$ or $|x - a| < \delta \Rightarrow |f(x)| < B$. Thus,

$$|x - a| < \delta \Rightarrow |f(x)g(x) - 0| = |f(x)g(x)|$$

$$= |f(x)||g(x)| < B \cdot \frac{\varepsilon}{B} = \varepsilon \text{ so } \lim_{x \rightarrow a} f(x)g(x) = 0.$$

29. Choose $\varepsilon > 0$. Since $\lim_{x \rightarrow a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
That is, for
 $a - \delta < x < a$ or $a < x < a + \delta$,
 $L - \varepsilon < f(x) < L + \varepsilon$.
Let $f(a) = A$,
 $M = \max\{|L - \varepsilon|, |L + \varepsilon|, |A|\}$, $c = a - \delta$,
 $d = a + \delta$. Then for x in (c, d) , $|f(x)| \leq M$, since
either $x = a$, in which case
 $|f(x)| = |f(a)| = |A| \leq M$ or $0 < |x - a| < \delta$ so
 $L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

30. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now
take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where
 $0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and
 $0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.
Thus, for $0 < |x - a| < \delta$,
 $L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.
Combine the inequalities and use the fact
that $f(x) \leq g(x)$ to get
 $L - \varepsilon < f(x) \leq g(x) < M + \varepsilon$ which leads to
 $L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.
However,
 $L - M = \alpha > 2\varepsilon$
which is a contradiction.
Thus $L \leq M$.

31. (b) and (c) are equivalent to the definition of limit.

32. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with
 $0 < |x - c| < \delta$ such that $|f(x) - L| > \varepsilon$.

33. a. $g(x) = \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\frac{x + 6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has
an asymptote at $x \approx 3.49$.

c. If $\delta \leq \frac{1}{4}$, then $2.75 < x < 3$
or $3 < x < 3.25$ and by graphing
 $y = |g(x)| = \left| \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6} \right|$
on the interval $[2.75, 3.25]$, we see that
 $0 < \left| \frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6} \right| < 3$
so m must be at least three.

1.3 Concepts Review

- 48
- 4
- $-8; -4 + 5c$
- 0

Problem Set 1.3

- $\lim_{x \rightarrow 1} (2x + 1) = 4$
 $= \lim_{x \rightarrow 1} 2x + \lim_{x \rightarrow 1} 1 = 3$
 $= 2 \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 1 = 2, 1$
 $= 2(1) + 1 = 3$
- $\lim_{x \rightarrow -1} (3x^2 - 1) = 5$
 $= \lim_{x \rightarrow -1} 3x^2 - \lim_{x \rightarrow -1} 1 = 3$
 $= 3 \lim_{x \rightarrow -1} x^2 - \lim_{x \rightarrow -1} 1 = 8$
 $= 3 \left(\lim_{x \rightarrow -1} x \right)^2 - \lim_{x \rightarrow -1} 1 = 2, 1$
 $= 3(-1)^2 - 1 = 2$
- $\lim_{x \rightarrow 0} [(2x + 1)(x - 3)] = 6$
 $= \lim_{x \rightarrow 0} (2x + 1) \cdot \lim_{x \rightarrow 0} (x - 3) = 4, 5$
 $= \left(\lim_{x \rightarrow 0} 2x + \lim_{x \rightarrow 0} 1 \right) \cdot \left(\lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} 3 \right) = 3$
 $= \left(2 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 1 \right) \cdot \left(\lim_{x \rightarrow 0} x - \lim_{x \rightarrow 0} 3 \right) = 2, 1$
 $= [2(0) + 1](0 - 3) = -3$
- $\lim_{x \rightarrow \sqrt{2}} [(2x^2 + 1)(7x^2 + 13)] = 6$
 $= \lim_{x \rightarrow \sqrt{2}} (2x^2 + 1) \cdot \lim_{x \rightarrow \sqrt{2}} (7x^2 + 13) = 4, 3$
 $= \left(2 \lim_{x \rightarrow \sqrt{2}} x^2 + \lim_{x \rightarrow \sqrt{2}} 1 \right) \cdot \left(7 \lim_{x \rightarrow \sqrt{2}} x^2 + \lim_{x \rightarrow \sqrt{2}} 13 \right) = 8, 1$
 $= \left[2 \left(\lim_{x \rightarrow \sqrt{2}} x \right)^2 + 1 \right] \left[7 \left(\lim_{x \rightarrow \sqrt{2}} x \right)^2 + 13 \right] = 2$
 $= [2(\sqrt{2})^2 + 1][7(\sqrt{2})^2 + 13] = 135$

<p>5. $\lim_{x \rightarrow 2} \frac{2x+1}{5-3x}$ 7</p> <p>$= \frac{\lim_{x \rightarrow 2} (2x+1)}{\lim_{x \rightarrow 2} (5-3x)}$ 4, 5</p> <p>$= \frac{\lim_{x \rightarrow 2} 2x + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} 5 - \lim_{x \rightarrow 2} 3x}$ 3, 1</p> <p>$= \frac{2 \lim_{x \rightarrow 2} x + 1}{5 - 3 \lim_{x \rightarrow 2} x}$ 2</p> <p>$= \frac{2(2)+1}{5-3(2)} = -5$</p>	<p>9. $\lim_{t \rightarrow -2} (2t^3 + 15)^{13}$ 8</p> <p>$= \left[\lim_{t \rightarrow -2} (2t^3 + 15) \right]^{13}$ 4, 3</p> <p>$= \left[2 \lim_{t \rightarrow -2} t^3 + \lim_{t \rightarrow -2} 15 \right]^{13}$ 8</p> <p>$= \left[2 \left(\lim_{t \rightarrow -2} t \right)^3 + \lim_{t \rightarrow -2} 15 \right]^{13}$ 2, 1</p> <p>$= [2(-2)^3 + 15]^{13} = -1$</p>
<p>6. $\lim_{x \rightarrow -3} \frac{4x^3+1}{7-2x^2}$ 7</p> <p>$= \frac{\lim_{x \rightarrow -3} (4x^3+1)}{\lim_{x \rightarrow -3} (7-2x^2)}$ 4, 5</p> <p>$= \frac{\lim_{x \rightarrow -3} 4x^3 + \lim_{x \rightarrow -3} 1}{\lim_{x \rightarrow -3} 7 - \lim_{x \rightarrow -3} 2x^2}$ 3, 1</p> <p>$= \frac{4 \lim_{x \rightarrow -3} x^3 + 1}{7 - 2 \lim_{x \rightarrow -3} x^2}$ 8</p> <p>$= \frac{4 \left(\lim_{x \rightarrow -3} x \right)^3 + 1}{7 - 2 \left(\lim_{x \rightarrow -3} x \right)^2}$ 2</p> <p>$= \frac{4(-3)^3 + 1}{7 - 2(-3)^2} = \frac{107}{11}$</p>	<p>10. $\lim_{w \rightarrow -2} \sqrt{-3w^3 + 7w^2}$ 9</p> <p>$= \sqrt{\lim_{w \rightarrow -2} (-3w^3 + 7w^2)}$ 4, 3</p> <p>$= \sqrt{-3 \lim_{w \rightarrow -2} w^3 + 7 \lim_{w \rightarrow -2} w^2}$ 8</p> <p>$= \sqrt{-3 \left(\lim_{w \rightarrow -2} w \right)^3 + 7 \left(\lim_{w \rightarrow -2} w \right)^2}$ 2</p> <p>$= \sqrt{-3(-2)^3 + 7(-2)^2} = 2\sqrt{13}$</p>
<p>7. $\lim_{x \rightarrow 3} \sqrt{3x-5}$ 9</p> <p>$= \sqrt{\lim_{x \rightarrow 3} (3x-5)}$ 5, 3</p> <p>$= \sqrt{3 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 5}$ 2, 1</p> <p>$= \sqrt{3(3) - 5} = 2$</p>	<p>11. $\lim_{y \rightarrow 2} \left(\frac{4y^3 + 8y}{y+4} \right)^{1/3}$ 9</p> <p>$= \left(\lim_{y \rightarrow 2} \frac{4y^3 + 8y}{y+4} \right)^{1/3}$ 7</p> <p>$= \left[\frac{\lim_{y \rightarrow 2} (4y^3 + 8y)}{\lim_{y \rightarrow 2} (y+4)} \right]^{1/3}$ 4, 3</p> <p>$= \left(\frac{4 \lim_{y \rightarrow 2} y^3 + 8 \lim_{y \rightarrow 2} y}{\lim_{y \rightarrow 2} y + \lim_{y \rightarrow 2} 4} \right)^{1/3}$ 8, 1</p> <p>$= \left[\frac{4 \left(\lim_{y \rightarrow 2} y \right)^3 + 8 \lim_{y \rightarrow 2} y}{\lim_{y \rightarrow 2} y + 4} \right]^{1/3}$ 2</p> <p>$= \left[\frac{4(2)^3 + 8(2)}{2+4} \right]^{1/3} = 2$</p>
<p>8. $\lim_{x \rightarrow -3} \sqrt{5x^2 + 2x}$ 9</p> <p>$= \sqrt{\lim_{x \rightarrow -3} (5x^2 + 2x)}$ 4, 3</p> <p>$= \sqrt{5 \lim_{x \rightarrow -3} x^2 + 2 \lim_{x \rightarrow -3} x}$ 8</p> <p>$= \sqrt{5 \left(\lim_{x \rightarrow -3} x \right)^2 + 2 \lim_{x \rightarrow -3} x}$ 2</p> <p>$= \sqrt{5(-3)^2 + 2(-3)} = \sqrt{39}$</p>	

$$12. \lim_{w \rightarrow 5} (2w^4 - 9w^3 + 19)^{-1/2}$$

$$= \lim_{w \rightarrow 5} \frac{1}{\sqrt{2w^4 - 9w^3 + 19}} \quad 7$$

$$= \frac{\lim_{w \rightarrow 5} 1}{\lim_{w \rightarrow 5} \sqrt{2w^4 - 9w^3 + 19}} \quad 1, 9$$

$$= \frac{1}{\sqrt{\lim_{w \rightarrow 5} (2w^4 - 9w^3 + 19)}} \quad 4,5$$

$$= \frac{1}{\sqrt{\lim_{w \rightarrow 5} 2w^4 - \lim_{w \rightarrow 5} 9w^3 + \lim_{w \rightarrow 5} 19}} \quad 1,3$$

$$= \frac{1}{\sqrt{2 \lim_{w \rightarrow 5} w^4 - 9 \lim_{w \rightarrow 5} w^3 + 19}} \quad 8$$

$$= \frac{1}{\sqrt{2 \left(\lim_{w \rightarrow 5} w \right)^4 - 9 \left(\lim_{w \rightarrow 5} w \right)^3 + 19}} \quad 2$$

$$= \frac{1}{\sqrt{2(5)^4 - 9(5)^3 + 19}}$$

$$= \frac{1}{\sqrt{144}} = \frac{1}{12}$$

$$13. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + 4} = \frac{\lim_{x \rightarrow 2} (x^2 - 4)}{\lim_{x \rightarrow 2} (x^2 + 4)} = \frac{4 - 4}{4 + 4} = 0$$

$$14. \lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-3)(x-2)}{(x-2)}$$

$$= \lim_{x \rightarrow 2} (x-3) = -1$$

$$15. \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(x-3)(x+1)}{(x+1)}$$

$$= \lim_{x \rightarrow -1} (x-3) = -4$$

$$16. \lim_{x \rightarrow -1} \frac{x^2 + x}{x^2 + 1} = \frac{\lim_{x \rightarrow -1} (x^2 + x)}{\lim_{x \rightarrow -1} (x^2 + 1)} = \frac{0}{2} = 0$$

$$17. \lim_{x \rightarrow -1} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x \rightarrow -1} \frac{x-3}{x+7}$$

$$= \frac{-1-3}{-1+7} = -\frac{2}{3}$$

$$18. \lim_{x \rightarrow 2} \frac{x^2 + 7x + 10}{x + 2} = \lim_{x \rightarrow 2} \frac{(x+2)(x+5)}{x+2}$$

$$= \lim_{x \rightarrow 2} (x+5) = 7$$

$$19. \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$$

$$20. \lim_{x \rightarrow -3} \frac{x^2 - 14x - 51}{x^2 - 4x - 21} = \lim_{x \rightarrow -3} \frac{(x+3)(x-17)}{(x+3)(x-7)}$$

$$= \lim_{x \rightarrow -3} \frac{x-17}{x-7} = \frac{-3-17}{-3-7} = 2$$

$$21. \lim_{u \rightarrow -2} \frac{u^2 - ux + 2u - 2x}{u^2 - u - 6} = \lim_{u \rightarrow -2} \frac{(u+2)(u-x)}{(u+2)(u-3)}$$

$$= \lim_{u \rightarrow -2} \frac{u-x}{u-3} = \frac{x+2}{5}$$

$$22. \lim_{x \rightarrow 1} \frac{x^2 + ux - x - u}{x^2 + 2x - 3} = \lim_{x \rightarrow 1} \frac{(x-1)(x+u)}{(x-1)(x+3)}$$

$$= \lim_{x \rightarrow 1} \frac{x+u}{x+3} = \frac{1+u}{1+3} = \frac{u+1}{4}$$

$$23. \lim_{x \rightarrow \pi} \frac{2x^2 - 6x\pi + 4\pi^2}{x^2 - \pi^2} = \lim_{x \rightarrow \pi} \frac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)}$$

$$= \lim_{x \rightarrow \pi} \frac{2(x-2\pi)}{x+\pi} = \frac{2(\pi-2\pi)}{\pi+\pi} = -1$$

$$24. \lim_{w \rightarrow -2} \frac{(w+2)(w^2 - w - 6)}{w^2 + 4w + 4}$$

$$= \lim_{w \rightarrow -2} \frac{(w+2)^2(w-3)}{(w+2)^2} = \lim_{w \rightarrow -2} (w-3)$$

$$= -2 - 3 = -5$$

$$25. \lim_{x \rightarrow a} \sqrt{f^2(x) + g^2(x)}$$

$$= \sqrt{\lim_{x \rightarrow a} f^2(x) + \lim_{x \rightarrow a} g^2(x)}$$

$$= \sqrt{\left(\lim_{x \rightarrow a} f(x) \right)^2 + \left(\lim_{x \rightarrow a} g(x) \right)^2}$$

$$= \sqrt{(3)^2 + (-1)^2} = \sqrt{10}$$

$$26. \lim_{x \rightarrow a} \frac{2f(x) - 3g(x)}{f(x) + g(x)} = \frac{\lim_{x \rightarrow a} [2f(x) - 3g(x)]}{\lim_{x \rightarrow a} [f(x) + g(x)]}$$

$$= \frac{2 \lim_{x \rightarrow a} f(x) - 3 \lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)} = \frac{2(3) - 3(-1)}{3 + (-1)} = \frac{9}{2}$$

27. $\lim_{x \rightarrow a} \sqrt[3]{g(x)[f(x)+3]} = \lim_{x \rightarrow a} \sqrt[3]{g(x)} \cdot \lim_{x \rightarrow a} [f(x)+3]$
 $= \sqrt[3]{\lim_{x \rightarrow a} g(x)} \cdot \left[\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} 3 \right] = \sqrt[3]{-1} \cdot (3+3)$
 $= -6$
28. $\lim_{x \rightarrow a} [f(x)-3]^4 = \left[\lim_{x \rightarrow a} (f(x)-3) \right]^4$
 $= \left[\lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} 3 \right]^4 = (3-3)^4 = 0$
29. $\lim_{t \rightarrow a} [|f(t)|+|3g(t)|] = \lim_{t \rightarrow a} |f(t)| + 3 \lim_{t \rightarrow a} |g(t)|$
 $= \left| \lim_{t \rightarrow a} f(t) \right| + 3 \left| \lim_{t \rightarrow a} g(t) \right|$
 $= |3| + 3|-1| = 6$
30. $\lim_{u \rightarrow a} [f(u)+3g(u)]^3 = \left(\lim_{u \rightarrow a} [f(u)+3g(u)] \right)^3$
 $= \left[\lim_{u \rightarrow a} f(u) + 3 \lim_{u \rightarrow a} g(u) \right]^3 = [3+3(-1)]^3 = 0$
31. $\lim_{x \rightarrow 2} \frac{3x^2-12}{x-2} = \lim_{x \rightarrow 2} \frac{3(x-2)(x+2)}{x-2}$
 $= 3 \lim_{x \rightarrow 2} (x+2) = 3(2+2) = 12$
32. $\lim_{x \rightarrow 2} \frac{(3x^2+2x+1)-17}{x-2} = \lim_{x \rightarrow 2} \frac{3x^2+2x-16}{x-2}$
 $= \lim_{x \rightarrow 2} \frac{(3x+8)(x-2)}{x-2} = \lim_{x \rightarrow 2} (3x+8)$
 $= 3 \lim_{x \rightarrow 2} x + 8 = 3(2) + 8 = 14$
33. $\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2} = \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x-2} = \lim_{x \rightarrow 2} \frac{-\frac{x-2}{2x}}{x-2}$
 $= \lim_{x \rightarrow 2} -\frac{1}{2x} = \frac{-1}{2 \lim_{x \rightarrow 2} x} = \frac{-1}{2(2)} = -\frac{1}{4}$
34. $\lim_{x \rightarrow 2} \frac{\frac{3}{x^2} - \frac{3}{4}}{x-2} = \lim_{x \rightarrow 2} \frac{\frac{3(4-x^2)}{4x^2}}{x-2} = \lim_{x \rightarrow 2} \frac{-3(x+2)(x-2)}{4x^2(x-2)}$
 $= \lim_{x \rightarrow 2} \frac{-3(x+2)}{4x^2} = \frac{-3 \left(\lim_{x \rightarrow 2} x + 2 \right)}{4 \left(\lim_{x \rightarrow 2} x \right)^2} = \frac{-3(2+2)}{4(2)^2}$
 $= -\frac{3}{4}$

35. Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$.
 $|f(x)g(x) - LM| \leq |g(x)||f(x) - L| + |L||g(x) - M|$
as shown in the text. Choose $\varepsilon_1 = 1$. Since
 $\lim_{x \rightarrow c} g(x) = M$, there is some $\delta_1 > 0$ such that if

$$0 < |x - c| < \delta_1, \quad |g(x) - M| < \varepsilon_1 = 1 \quad \text{or}$$

$$M - 1 < g(x) < M + 1$$

$$|M - 1| \leq |M| + 1 \quad \text{and} \quad |M + 1| \leq |M| + 1 \quad \text{so for}$$

$$0 < |x - c| < \delta_1, \quad |g(x)| < |M| + 1. \quad \text{Choose } \varepsilon > 0.$$

Since $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, there

exist δ_2 and δ_3 such that $0 < |x - c| < \delta_2 \Rightarrow$

$$|f(x) - L| < \frac{\varepsilon}{|L| + |M| + 1} \quad \text{and} \quad 0 < |x - c| < \delta_3 \Rightarrow$$

$$|g(x) - M| < \frac{\varepsilon}{|L| + |M| + 1}. \quad \text{Let}$$

$\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $0 < |x - c| < \delta \Rightarrow$

$$|f(x)g(x) - LM| \leq |g(x)||f(x) - L| + |L||g(x) - M|$$

$$< (|M| + 1) \frac{\varepsilon}{|L| + |M| + 1} + |L| \frac{\varepsilon}{|L| + |M| + 1} = \varepsilon$$

Hence,

$$\lim_{x \rightarrow c} f(x)g(x) = LM = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right)$$

36. Say $\lim_{x \rightarrow c} g(x) = M$, $M \neq 0$, and choose

$$\varepsilon_1 = \frac{1}{2}|M|.$$

There is some $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |g(x) - M| < \varepsilon_1 = \frac{1}{2}|M| \quad \text{or}$$

$$M - \frac{1}{2}|M| < g(x) < M + \frac{1}{2}|M|.$$

$$\left| M - \frac{1}{2}|M| \right| \geq \left| \frac{1}{2}|M| \right| \quad \text{and} \quad \left| M + \frac{1}{2}|M| \right| \geq \left| \frac{1}{2}|M| \right|$$

$$\text{so } |g(x)| > \frac{1}{2}|M| \quad \text{and} \quad \frac{1}{|g(x)|} < \frac{2}{|M|}$$

Choose $\varepsilon > 0$.

Since $\lim_{x \rightarrow c} g(x) = M$ there is $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{1}{2}M^2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, then

$$0 < |x - c| < \delta \Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x)M} \right|$$

$$= \frac{1}{|M||g(x)|} |g(x) - M| < \frac{2}{M^2} |g(x) - M| = \frac{2}{M^2} \cdot \frac{1}{2} M^2 \varepsilon$$

$= \varepsilon$

Thus, $\lim_{x \rightarrow c} \frac{1}{g(x)} = \frac{1}{M} = \frac{1}{\lim_{x \rightarrow c} g(x)}$.

Using statement 6 and the above result,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} \frac{1}{g(x)} \\ &= \lim_{x \rightarrow c} f(x) \cdot \frac{1}{\lim_{x \rightarrow c} g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}. \end{aligned}$$

37. $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} L$
 $\Leftrightarrow \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} L = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} [f(x) - L] = 0$

38. $\lim_{x \rightarrow c} f(x) = 0 \Leftrightarrow \left[\lim_{x \rightarrow c} f(x) \right]^2 = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} f^2(x) = 0$
 $\Leftrightarrow \sqrt{\lim_{x \rightarrow c} f^2(x)} = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} \sqrt{f^2(x)} = 0$
 $\Leftrightarrow \lim_{x \rightarrow c} |f(x)| = 0$

39. $\lim_{x \rightarrow c} |x| = \sqrt{\left(\lim_{x \rightarrow c} |x| \right)^2} = \sqrt{\lim_{x \rightarrow c} |x|^2} = \sqrt{\lim_{x \rightarrow c} x^2}$
 $= \sqrt{\left(\lim_{x \rightarrow c} x \right)^2} = \sqrt{c^2} = |c|$

40. a. If $f(x) = \frac{x+1}{x-2}$, $g(x) = \frac{x-5}{x-2}$ and $c = 2$, then
 $\lim_{x \rightarrow c} [f(x) + g(x)]$ exists, but neither
 $\lim_{x \rightarrow c} f(x)$ nor $\lim_{x \rightarrow c} g(x)$ exists.

b. If $f(x) = \frac{2}{x}$, $g(x) = x$, and $c = 0$, then
 $\lim_{x \rightarrow c} [f(x) \cdot g(x)]$ exists, but $\lim_{x \rightarrow c} f(x)$ does
not exist.

41. $\lim_{x \rightarrow -3^+} \frac{\sqrt{3+x}}{x} = \frac{\sqrt{3-3}}{-3} = 0$

42. $\lim_{x \rightarrow -\pi^+} \frac{\sqrt{\pi^3 + x^3}}{x} = \frac{\sqrt{\pi^3 + (-\pi)^3}}{-\pi} = 0$

43. $\lim_{x \rightarrow 3^+} \frac{x-3}{\sqrt{x^2-9}} = \lim_{x \rightarrow 3^+} \frac{(x-3)\sqrt{x^2-9}}{x^2-9}$
 $= \lim_{x \rightarrow 3^+} \frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)} = \lim_{x \rightarrow 3^+} \frac{\sqrt{x^2-9}}{x+3}$
 $= \frac{\sqrt{3^2-9}}{3+3} = 0$

44. $\lim_{x \rightarrow 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{1+1}}{4+4(1)} = \frac{\sqrt{2}}{8}$

45. $\lim_{x \rightarrow 2^+} \frac{(x^2+1)\llbracket x \rrbracket}{(3x-1)^2} = \frac{(2^2+1)\llbracket 2 \rrbracket}{(3 \cdot 2-1)^2} = \frac{5 \cdot 2}{5^2} = \frac{2}{5}$

46. $\lim_{x \rightarrow 3^-} (x - \llbracket x \rrbracket) = \lim_{x \rightarrow 3^-} x - \lim_{x \rightarrow 3^-} \llbracket x \rrbracket = 3 - 2 = 1$

47. $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$

48. $\lim_{x \rightarrow 3^+} \llbracket x^2 + 2x \rrbracket = \llbracket 3^2 + 2 \cdot 3 \rrbracket = 15$

49. $f(x)g(x) = 1$; $g(x) = \frac{1}{f(x)}$
 $\lim_{x \rightarrow a} g(x) = 0 \Leftrightarrow \lim_{x \rightarrow a} \frac{1}{f(x)} = 0$
 $\Leftrightarrow \frac{1}{\lim_{x \rightarrow a} f(x)} = 0$

No value satisfies this equation, so $\lim_{x \rightarrow a} f(x)$
must not exist.

50. R has the vertices $\left(\pm \frac{x}{2}, \pm \frac{1}{2} \right)$

Each side of Q has length $\sqrt{x^2+1}$ so the
perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of
length 1 and two sides of length $\sqrt{x^2}$ so the
perimeter of R is $2+2\sqrt{x^2}$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q} &= \lim_{x \rightarrow 0^+} \frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}} \\ &= \frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
 51. \text{ a. } NO &= \sqrt{(0-0)^2 + (1-0)^2} = 1 \\
 OP &= \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} \\
 &= \sqrt{x^2 + x} \\
 NP &= \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + y^2 - 2y + 1} \\
 &= \sqrt{x^2 + x - 2\sqrt{x} + 1} \\
 MO &= \sqrt{(1-0)^2 + (0-0)^2} = 1 \\
 MP &= \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{y^2 + x^2 - 2x + 1} \\
 &= \sqrt{x^2 - x + 1} \\
 \lim_{x \rightarrow 0^+} \frac{\text{perimeter of } \triangle NOP}{\text{perimeter of } \triangle MOP} &= \lim_{x \rightarrow 0^+} \frac{1 + \sqrt{x^2 + x} + \sqrt{x^2 + x - 2\sqrt{x} + 1}}{1 + \sqrt{x^2 + x} + \sqrt{x^2 - x + 1}} \\
 &= \frac{1 + \sqrt{1}}{1 + \sqrt{1}} = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b. Area of } \triangle NOP &= \frac{1}{2}(1)(x) = \frac{x}{2} \\
 \text{Area of } \triangle MOP &= \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2} \\
 \lim_{x \rightarrow 0^+} \frac{\text{area of } \triangle NOP}{\text{area of } \triangle MOP} &= \lim_{x \rightarrow 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x \rightarrow 0^+} \frac{x}{\sqrt{x}} \\
 &= \lim_{x \rightarrow 0^+} \sqrt{x} = 0
 \end{aligned}$$

1.4 Concepts Review

- 0
- 1
- the denominator is 0 when $t = 0$.
- 1

Problem Set 1.4

- $\lim_{x \rightarrow 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$
- $\lim_{\theta \rightarrow \pi/2} \theta \cos \theta = \frac{\pi}{2} \cdot 0 = 0$
- $\lim_{t \rightarrow 0} \frac{\cos^2 t}{1 + \sin t} = \frac{\cos^2 0}{1 + \sin 0} = \frac{1}{1+0} = 1$

$$\begin{aligned}
 4. \lim_{x \rightarrow 0} \frac{3x \tan x}{\sin x} &= \lim_{x \rightarrow 0} \frac{3x(\sin x / \cos x)}{\sin x} = \lim_{x \rightarrow 0} \frac{3x}{\cos x} \\
 &= \frac{0}{1} = 0
 \end{aligned}$$

$$5. \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$\begin{aligned}
 6. \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{2\theta} &= \lim_{\theta \rightarrow 0} \frac{3}{2} \cdot \frac{\sin 3\theta}{3\theta} = \frac{3}{2} \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \\
 &= \frac{3}{2} \cdot 1 = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 7. \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\tan \theta} &= \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\frac{\sin \theta}{\cos \theta}} = \lim_{\theta \rightarrow 0} \frac{\cos \theta \sin 3\theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \left[\cos \theta \cdot 3 \cdot \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \right] \\
 &= 3 \lim_{\theta \rightarrow 0} \left[\cos \theta \cdot \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\frac{\sin \theta}{\theta}} \right] = 3 \cdot 1 \cdot 1 \cdot 1 = 3
 \end{aligned}$$

$$\begin{aligned}
 8. \lim_{\theta \rightarrow 0} \frac{\tan 5\theta}{\sin 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{\sin 5\theta}{\cos 5\theta \sin 2\theta} \\
 &= \lim_{\theta \rightarrow 0} \left[\frac{1}{\cos 5\theta} \cdot 5 \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{1}{2} \cdot \frac{2\theta}{\sin 2\theta} \right] \\
 &= \frac{5}{2} \lim_{\theta \rightarrow 0} \left[\frac{1}{\cos 5\theta} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{2\theta}{\sin 2\theta} \right] \\
 &= \frac{5}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 9. \lim_{\theta \rightarrow 0} \frac{\cot \pi \theta \sin \theta}{2 \sec \theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\cos \pi \theta}{\sin \pi \theta} \sin \theta}{\frac{2}{\cos \theta}} \\
 &= \lim_{\theta \rightarrow 0} \frac{\cos \pi \theta \sin \theta \cos \theta}{2 \sin \pi \theta} \\
 &= \lim_{\theta \rightarrow 0} \left[\frac{\cos \pi \theta \cos \theta}{2} \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\pi} \cdot \frac{\pi \theta}{\sin \pi \theta} \right] \\
 &= \frac{1}{2\pi} \lim_{\theta \rightarrow 0} \left[\cos \pi \theta \cos \theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{\pi \theta}{\sin \pi \theta} \right] \\
 &= \frac{1}{2\pi} \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \frac{1}{2\pi}
 \end{aligned}$$

$$10. \lim_{t \rightarrow 0} \frac{\sin^2 3t}{2t} = \lim_{t \rightarrow 0} \frac{9t}{2} \cdot \frac{\sin 3t}{3t} \cdot \frac{\sin 3t}{3t} = 0 \cdot 1 \cdot 1 = 0$$

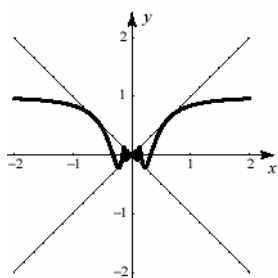
$$\begin{aligned}
 11. \lim_{t \rightarrow 0} \frac{\tan^2 3t}{2t} &= \lim_{t \rightarrow 0} \frac{\sin^2 3t}{(2t)(\cos^2 3t)} \\
 &= \lim_{t \rightarrow 0} \frac{3(\sin 3t)}{2 \cos^2 3t} \cdot \frac{\sin 3t}{3t} = 0 \cdot 1 = 0
 \end{aligned}$$

$$12. \lim_{t \rightarrow 0} \frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$$

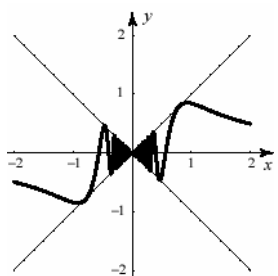
$$\begin{aligned}
 13. \quad \lim_{t \rightarrow 0} \frac{\sin(3t) + 4t}{t \sec t} &= \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{t \sec t} + \frac{4t}{t \sec t} \right) \\
 &= \lim_{t \rightarrow 0} \frac{\sin 3t}{t \sec t} + \lim_{t \rightarrow 0} \frac{4t}{t \sec t} \\
 &= \lim_{t \rightarrow 0} 3 \cos t \cdot \frac{\sin 3t}{3t} + \lim_{t \rightarrow 0} 4 \cos t \\
 &= 3 \cdot 1 + 4 = 7
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta^2} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta \sin \theta}{\theta \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \times \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \times 1 = 1
 \end{aligned}$$

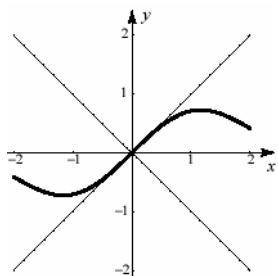
$$15. \quad \lim_{x \rightarrow 0} x \sin(1/x) = 0$$



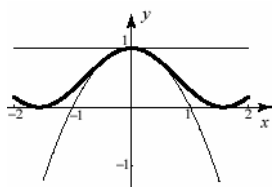
$$16. \quad \lim_{x \rightarrow 0} x \sin(1/x^2) = 0$$



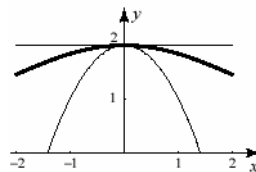
$$17. \quad \lim_{x \rightarrow 0} (1 - \cos^2 x) / x = 0$$



$$18. \quad \lim_{x \rightarrow 0} \cos^2 x = 1$$



$$19. \quad \lim_{x \rightarrow 0} 1 + \frac{\sin x}{x} = 2$$



20. The result that $\lim_{t \rightarrow 0} \cos t = 1$ was established in

the proof of the theorem. Then

$$\begin{aligned}
 \lim_{t \rightarrow c} \cos t &= \lim_{h \rightarrow 0} \cos(c+h) \\
 &= \lim_{h \rightarrow 0} (\cos c \cos h - \sin c \sin h) \\
 &= \lim_{h \rightarrow 0} \cos c \lim_{h \rightarrow 0} \cos h - \sin c \lim_{h \rightarrow 0} \sin h \\
 &= \cos c
 \end{aligned}$$

$$21. \quad \lim_{t \rightarrow c} \tan t = \lim_{t \rightarrow c} \frac{\sin t}{\cos t} = \frac{\lim_{t \rightarrow c} \sin t}{\lim_{t \rightarrow c} \cos t} = \frac{\sin c}{\cos c} = \tan c$$

$$\lim_{t \rightarrow c} \cot t = \lim_{t \rightarrow c} \frac{\cos t}{\sin t} = \frac{\lim_{t \rightarrow c} \cos t}{\lim_{t \rightarrow c} \sin t} = \frac{\cos c}{\sin c} = \cot c$$

$$22. \quad \lim_{t \rightarrow c} \sec t = \lim_{t \rightarrow c} \frac{1}{\cos t} = \frac{1}{\cos c} = \sec c$$

$$\lim_{t \rightarrow c} \csc t = \lim_{t \rightarrow c} \frac{1}{\sin t} = \frac{1}{\sin c} = \csc c$$

$$23. \quad \overline{BP} = \sin t, \overline{OB} = \cos t$$

$$\text{area}(\triangle OBP) \leq \text{area}(\text{sector } OAP)$$

$$\leq \text{area}(\triangle OBP) + \text{area}(ABPQ)$$

$$\frac{1}{2} \overline{OB} \cdot \overline{BP} \leq \frac{1}{2} t(1)^2 \leq \frac{1}{2} \overline{OB} \cdot \overline{BP} + (1 - \overline{OB}) \overline{BP}$$

$$\frac{1}{2} \sin t \cos t \leq \frac{1}{2} t \leq \frac{1}{2} \sin t \cos t + (1 - \cos t) \sin t$$

$$\cos t \leq \frac{t}{\sin t} \leq 2 - \cos t$$

$$\frac{1}{2 - \cos t} \leq \frac{\sin t}{t} \leq \frac{1}{\cos t} \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\lim_{t \rightarrow 0} \frac{1}{2 - \cos t} \leq \lim_{t \rightarrow 0} \frac{\sin t}{t} \leq \lim_{t \rightarrow 0} \frac{1}{\cos t}$$

$$1 \leq \lim_{t \rightarrow 0} \frac{\sin t}{t} \leq 1$$

$$\text{Thus, } \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1.$$

24. a. Written response

$$\begin{aligned} \text{b. } D &= \frac{1}{2} \overline{AB} \cdot \overline{BP} = \frac{1}{2}(1 - \cos t) \sin t \\ &= \frac{\sin t(1 - \cos t)}{2} \\ E &= \frac{1}{2}t(1)^2 - \frac{1}{2} \overline{OB} \cdot \overline{BP} = \frac{t}{2} - \frac{\sin t \cos t}{2} \\ \frac{D}{E} &= \frac{\sin t(1 - \cos t)}{t - \sin t \cos t} \end{aligned}$$

$$\text{c. } \lim_{t \rightarrow 0^+} \left(\frac{D}{E} \right) = 0.75$$

$$6. \lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 8x + 15} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{8}{x} + \frac{15}{x^2}} = 1$$

$$7. \lim_{x \rightarrow \infty} \frac{x^3}{2x^3 - 100x^2} = \lim_{x \rightarrow \infty} \frac{1}{2 - \frac{100}{x}} = \frac{1}{2}$$

$$8. \lim_{\theta \rightarrow -\infty} \frac{\pi \theta^5}{\theta^5 - 5\theta^4} = \lim_{\theta \rightarrow -\infty} \frac{\pi}{1 - \frac{5}{\theta}} = \pi$$

$$9. \lim_{x \rightarrow \infty} \frac{3x^3 - x^2}{\pi x^3 - 5x^2} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x}}{\pi - \frac{5}{x}} = \frac{3}{\pi}$$

$$10. \lim_{\theta \rightarrow \infty} \frac{\sin^2 \theta}{\theta^2 - 5}; 0 \leq \sin^2 \theta \leq 1 \text{ for all } \theta \text{ and}$$

$$\lim_{\theta \rightarrow \infty} \frac{1}{\theta^2 - 5} = \lim_{\theta \rightarrow \infty} \frac{\frac{1}{\theta^2}}{1 - \frac{5}{\theta^2}} = 0 \text{ so } \lim_{\theta \rightarrow \infty} \frac{\sin^2 \theta}{\theta^2 - 5} = 0$$

$$11. \lim_{x \rightarrow \infty} \frac{3\sqrt{x^3} + 3x}{\sqrt{2x^3}} = \lim_{x \rightarrow \infty} \frac{3x^{3/2} + 3x}{\sqrt{2}x^{3/2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3 + \frac{3}{\sqrt{x}}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$12. \lim_{x \rightarrow \infty} \sqrt[3]{\frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}}$$

$$= \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\pi + \frac{3}{x^2}}{\sqrt{2} + \frac{7}{x^2}}} = \sqrt[3]{\frac{\pi}{\sqrt{2}}}$$

$$13. \lim_{x \rightarrow \infty} \sqrt[3]{\frac{1 + 8x^2}{x^2 + 4}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{1 + 8x^2}{x^2 + 4}}$$

$$= \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} + 8}{1 + \frac{4}{x^2}}} = \sqrt[3]{8} = 2$$

$$14. \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 + x + 3}{(x-1)(x+1)}} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^2 + x + 3}{x^2 - 1}}$$

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{1}{x^2}}} = \sqrt{1} = 1$$

$$15. \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2}$$

1.5 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound
2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left
3. $y = 6$; horizontal
4. $x = 6$; vertical

Problem Set 1.5

$$1. \lim_{x \rightarrow \infty} \frac{x}{x-5} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{5}{x}} = 1$$

$$2. \lim_{x \rightarrow \infty} \frac{x^2}{5-x^3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{5}{x^3} - 1} = 0$$

$$3. \lim_{t \rightarrow -\infty} \frac{t^2}{7-t^2} = \lim_{t \rightarrow -\infty} \frac{1}{\frac{7}{t^2} - 1} = -1$$

$$4. \lim_{t \rightarrow -\infty} \frac{t}{t-5} = \lim_{t \rightarrow -\infty} \frac{1}{1 - \frac{5}{t}} = 1$$

$$5. \lim_{x \rightarrow \infty} \frac{x^2}{(x-5)(3-x)} = \lim_{x \rightarrow \infty} \frac{x^2}{-x^2 + 8x - 15}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{-1 + \frac{8}{x} - \frac{15}{x^2}} = -1$$

$$16. \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = \frac{1}{1 + 0} = 1$$

$$17. \lim_{n \rightarrow \infty} \frac{n^2}{n+1} = \lim_{n \rightarrow \infty} \frac{n}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} n}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)} = \frac{\infty}{1 + 0} = \infty$$

$$18. \lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = \frac{0}{1 + 0} = 0$$

19. For $x > 0$, $x = \sqrt{x^2}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x+1}{\sqrt{x^2+3}} &= \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x}}{\sqrt{\frac{x^2+3}{x^2}}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x^2}}} \\ &= \frac{2}{\sqrt{1}} = 2 \end{aligned}$$

$$20. \lim_{x \rightarrow \infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1 + \frac{4}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{2}{x} + \frac{1}{x^2}}}{1 + \frac{4}{x}} = 0$$

$$\begin{aligned} 21. \lim_{x \rightarrow \infty} \left(\sqrt{2x^2+3} - \sqrt{2x^2-5} \right) &= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{2x^2+3} - \sqrt{2x^2-5} \right) \left(\sqrt{2x^2+3} + \sqrt{2x^2-5} \right)}{\sqrt{2x^2+3} + \sqrt{2x^2-5}} \\ &= \lim_{x \rightarrow \infty} \frac{2x^2+3 - (2x^2-5)}{\sqrt{2x^2+3} + \sqrt{2x^2-5}} \\ &= \lim_{x \rightarrow \infty} \frac{8}{\sqrt{2x^2+3} + \sqrt{2x^2-5}} = \lim_{x \rightarrow \infty} \frac{\frac{8}{x}}{\sqrt{\frac{2x^2+3}{x^2}} + \sqrt{\frac{2x^2-5}{x^2}}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{8}{x}}{\sqrt{2 + \frac{3}{x^2}} + \sqrt{2 - \frac{5}{x^2}}} = 0 \end{aligned}$$

22.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\sqrt{x^2+2x} - x \right) &= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x^2+2x} - x \right) \left(\sqrt{x^2+2x} + x \right)}{\sqrt{x^2+2x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x} + x} = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2+2x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} = \frac{2}{2} = 1 \end{aligned}$$

$$23. \lim_{y \rightarrow -\infty} \frac{9y^3+1}{y^2-2y+2} = \lim_{y \rightarrow -\infty} \frac{9y + \frac{1}{y^2}}{1 - \frac{2}{y} + \frac{2}{y^2}} = -\infty$$

$$\begin{aligned} 24. \lim_{x \rightarrow \infty} \frac{a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n} \\ = \lim_{x \rightarrow \infty} \frac{a_0 + \frac{a_1}{x} + \dots + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}}{b_0 + \frac{b_1}{x} + \dots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}} = \frac{a_0}{b_0} \end{aligned}$$

$$25. \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

$$26. \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^3+2n+1}} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1 + \frac{2}{n^2} + \frac{1}{n^3}}} = \frac{\infty}{1} = \infty$$

27. As $x \rightarrow 4^+$, $x \rightarrow 4$ while $x-4 \rightarrow 0^+$.

$$\lim_{x \rightarrow 4^+} \frac{x}{x-4} = \infty$$

$$\begin{aligned} 28. \lim_{t \rightarrow -3^+} \frac{t^2-9}{t+3} &= \lim_{t \rightarrow -3^+} \frac{(t+3)(t-3)}{t+3} \\ &= \lim_{t \rightarrow -3^+} (t-3) = -6 \end{aligned}$$

29. As $t \rightarrow 3^-$, $t^2 \rightarrow 9$ while $9-t^2 \rightarrow 0^+$.

$$\lim_{t \rightarrow 3^-} \frac{t^2}{9-t^2} = \infty$$

30. As $x \rightarrow \sqrt[3]{5}^+$, $x^2 \rightarrow 5^{2/3}$ while $5-x^3 \rightarrow 0^-$.

$$\lim_{x \rightarrow \sqrt[3]{5}^+} \frac{x^2}{5-x^3} = -\infty$$

31. As $x \rightarrow 5^-$, $x^2 \rightarrow 25$, $x-5 \rightarrow 0^-$, and $3-x \rightarrow -2$.

$$\lim_{x \rightarrow 5^-} \frac{x^2}{(x-5)(3-x)} = \infty$$

32. As $\theta \rightarrow \pi^+$, $\theta^2 \rightarrow \pi^2$ while $\sin \theta \rightarrow 0^-$.

$$\lim_{\theta \rightarrow \pi^+} \frac{\theta^2}{\sin \theta} = -\infty$$

33. As $x \rightarrow 3^-$, $x^3 \rightarrow 27$, while $x - 3 \rightarrow 0^-$.

$$\lim_{x \rightarrow 3^-} \frac{x^3}{x-3} = -\infty$$

34. As $\theta \rightarrow \frac{\pi^+}{2}$, $\pi\theta \rightarrow \frac{\pi^2}{2}$ while $\cos \theta \rightarrow 0^-$.

$$\lim_{\theta \rightarrow \frac{\pi^+}{2}} \frac{\pi\theta}{\cos \theta} = -\infty$$

35.
$$\lim_{x \rightarrow 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3^-} \frac{(x+2)(x-3)}{x-3}$$

$$= \lim_{x \rightarrow 3^-} (x+2) = 5$$

36.
$$\lim_{x \rightarrow 2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \rightarrow 2^+} \frac{(x+4)(x-2)}{(x+2)(x-2)}$$

$$= \lim_{x \rightarrow 2^+} \frac{x+4}{x+2} = \frac{6}{4} = \frac{3}{2}$$

37. For $0 \leq x < 1$, $\llbracket x \rrbracket = 0$, so for $0 < x < 1$, $\frac{\llbracket x \rrbracket}{x} = 0$

thus
$$\lim_{x \rightarrow 0^+} \frac{\llbracket x \rrbracket}{x} = 0$$

38. For $-1 \leq x < 0$, $\llbracket x \rrbracket = -1$, so for $-1 < x < 0$,

$$\frac{\llbracket x \rrbracket}{x} = -\frac{1}{x} \text{ thus } \lim_{x \rightarrow 0^-} \frac{\llbracket x \rrbracket}{x} = \infty.$$

(Since $x < 0$, $-\frac{1}{x} > 0$.)

39. For $x < 0$, $|x| = -x$, thus

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

40. For $x > 0$, $|x| = x$, thus $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$

41. As $x \rightarrow 0^-$, $1 + \cos x \rightarrow 2$ while $\sin x \rightarrow 0^-$.

$$\lim_{x \rightarrow 0^-} \frac{1 + \cos x}{\sin x} = -\infty$$

42. $-1 \leq \sin x \leq 1$ for all x , and

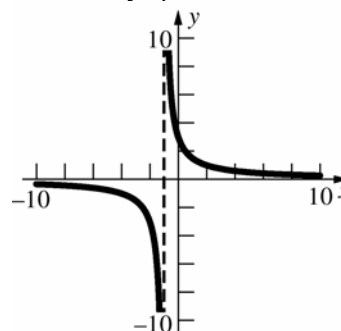
$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \text{ so } \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

43. $\lim_{x \rightarrow \infty} \frac{3}{x+1} = 0$, $\lim_{x \rightarrow -\infty} \frac{3}{x+1} = 0$;

Horizontal asymptote $y = 0$.

$$\lim_{x \rightarrow -1^+} \frac{3}{x+1} = \infty, \lim_{x \rightarrow -1^-} \frac{3}{x+1} = -\infty;$$

Vertical asymptote $x = -1$

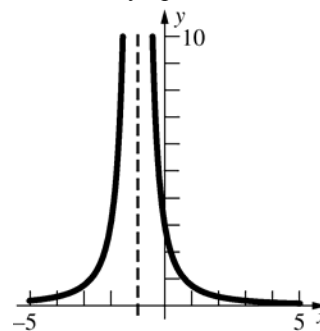


44. $\lim_{x \rightarrow \infty} \frac{3}{(x+1)^2} = 0$, $\lim_{x \rightarrow -\infty} \frac{3}{(x+1)^2} = 0$;

Horizontal asymptote $y = 0$.

$$\lim_{x \rightarrow -1^+} \frac{3}{(x+1)^2} = \infty, \lim_{x \rightarrow -1^-} \frac{3}{(x+1)^2} = \infty;$$

Vertical asymptote $x = -1$



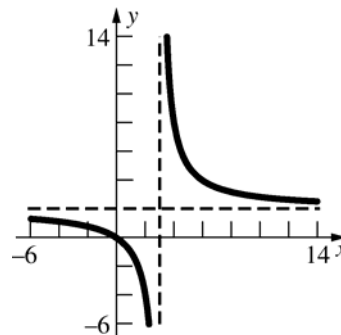
45. $\lim_{x \rightarrow \infty} \frac{2x}{x-3} = \lim_{x \rightarrow \infty} \frac{2}{1 - \frac{3}{x}} = 2$,

$$\lim_{x \rightarrow -\infty} \frac{2x}{x-3} = \lim_{x \rightarrow -\infty} \frac{2}{1 - \frac{3}{x}} = 2,$$

Horizontal asymptote $y = 2$

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty, \lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty;$$

Vertical asymptote $x = 3$



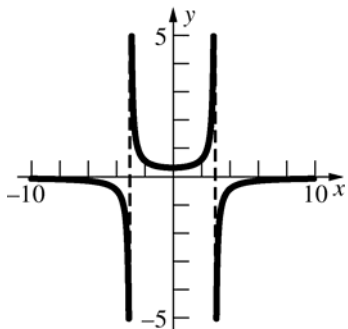
$$46. \lim_{x \rightarrow \infty} \frac{3}{9-x^2} = 0, \lim_{x \rightarrow -\infty} \frac{3}{9-x^2} = 0;$$

Horizontal asymptote $y = 0$

$$\lim_{x \rightarrow 3^+} \frac{3}{9-x^2} = -\infty, \lim_{x \rightarrow 3^-} \frac{3}{9-x^2} = \infty,$$

$$\lim_{x \rightarrow -3^+} \frac{3}{9-x^2} = \infty, \lim_{x \rightarrow -3^-} \frac{3}{9-x^2} = -\infty;$$

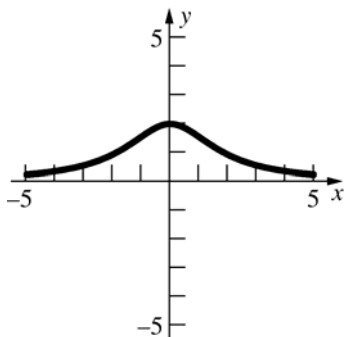
Vertical asymptotes $x = -3, x = 3$



$$47. \lim_{x \rightarrow \infty} \frac{14}{2x^2+7} = 0, \lim_{x \rightarrow -\infty} \frac{14}{2x^2+7} = 0;$$

Horizontal asymptote $y = 0$

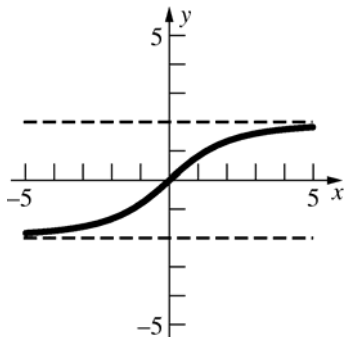
Since $2x^2 + 7 > 0$ for all x , $g(x)$ has no vertical asymptotes.



$$48. \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1+\frac{5}{x^2}}} = \frac{2}{\sqrt{1}} = 2,$$

$$\lim_{x \rightarrow -\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x \rightarrow -\infty} \frac{2}{-\sqrt{1+\frac{5}{x^2}}} = \frac{2}{-\sqrt{1}} = -2$$

Since $\sqrt{x^2+5} > 0$ for all x , $g(x)$ has no vertical asymptotes.



$$49. f(x) = 2x + 3 - \frac{1}{x^3-1}, \text{ thus}$$

$$\lim_{x \rightarrow \infty} [f(x) - (2x+3)] = \lim_{x \rightarrow \infty} \left[-\frac{1}{x^3-1} \right] = 0$$

The oblique asymptote is $y = 2x + 3$.

$$50. f(x) = 3x + 4 - \frac{4x+3}{x^2+1}, \text{ thus}$$

$$\lim_{x \rightarrow \infty} [f(x) - (3x+4)] = \lim_{x \rightarrow \infty} \left[-\frac{4x+3}{x^2+1} \right]$$

$$= \lim_{x \rightarrow \infty} \left[-\frac{\frac{4}{x} + \frac{3}{x^2}}{1 + \frac{1}{x^2}} \right] = 0.$$

The oblique asymptote is $y = 3x + 4$.

$$51. \text{ a. We say that } \lim_{x \rightarrow c^+} f(x) = -\infty \text{ if to each}$$

negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

$$\text{ b. We say that } \lim_{x \rightarrow c^-} f(x) = \infty \text{ if to each}$$

positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

$$52. \text{ a. We say that } \lim_{x \rightarrow \infty} f(x) = \infty \text{ if to each}$$

positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

$$\text{ b. We say that } \lim_{x \rightarrow -\infty} f(x) = \infty \text{ if to each}$$

positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

$$53. \text{ Let } \varepsilon > 0 \text{ be given. Since } \lim_{x \rightarrow \infty} f(x) = A, \text{ there is}$$

a corresponding number M_1 such that

$$x > M_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}. \text{ Similarly, there is a}$$

$$\text{number } M_2 \text{ such that } x > M_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}.$$

Let $M = \max\{M_1, M_2\}$, then

$$x > M \Rightarrow |f(x) + g(x) - (A+B)|$$

$$= |f(x) - A + g(x) - B| \leq |f(x) - A| + |g(x) - B|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\text{Thus, } \lim_{x \rightarrow \infty} [f(x) + g(x)] = A + B$$

54. Written response

- 55. a.** $\lim_{x \rightarrow \infty} \sin x$ does not exist as $\sin x$ oscillates between -1 and 1 as x increases.
- b.** Let $u = \frac{1}{x}$, then as $x \rightarrow \infty, u \rightarrow 0^+$.
- $$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{u \rightarrow 0^+} \sin u = 0$$
- c.** Let $u = \frac{1}{x}$, then as $x \rightarrow \infty, u \rightarrow 0^+$.
- $$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{u \rightarrow 0^+} \frac{1}{u} \sin u = \lim_{u \rightarrow 0^+} \frac{\sin u}{u} = 1$$
- d.** Let $u = \frac{1}{x}$, then
- $$\begin{aligned} \lim_{x \rightarrow \infty} x^{3/2} \sin \frac{1}{x} &= \lim_{u \rightarrow 0^+} \left(\frac{1}{u} \right)^{3/2} \sin u \\ &= \lim_{u \rightarrow 0^+} \left[\left(\frac{1}{\sqrt{u}} \right) \left(\frac{\sin u}{u} \right) \right] = \infty \end{aligned}$$
- e.** As $x \rightarrow \infty$, $\sin x$ oscillates between -1 and 1 , while $x^{-1/2} = \frac{1}{\sqrt{x}} \rightarrow 0$.
- $$\lim_{x \rightarrow \infty} x^{-1/2} \sin x = 0$$
- f.** Let $u = \frac{1}{x}$, then
- $$\begin{aligned} \lim_{x \rightarrow \infty} \sin \left(\frac{\pi}{6} + \frac{1}{x} \right) &= \lim_{u \rightarrow 0^+} \sin \left(\frac{\pi}{6} + u \right) \\ &= \sin \frac{\pi}{6} = \frac{1}{2} \end{aligned}$$
- g.** As $x \rightarrow \infty, x + \frac{1}{x} \rightarrow \infty$, so $\lim_{x \rightarrow \infty} \sin \left(x + \frac{1}{x} \right)$ does not exist. (See part a.)
- h.** $\sin \left(x + \frac{1}{x} \right) = \sin x \cos \frac{1}{x} + \cos x \sin \frac{1}{x}$
- $$\begin{aligned} \lim_{x \rightarrow \infty} \left[\sin \left(x + \frac{1}{x} \right) - \sin x \right] \\ &= \lim_{x \rightarrow \infty} \left[\sin x \left(\cos \frac{1}{x} - 1 \right) + \cos x \sin \frac{1}{x} \right] \end{aligned}$$
- As $x \rightarrow \infty, \cos \frac{1}{x} \rightarrow 1$ so $\cos \frac{1}{x} - 1 \rightarrow 0$.
- From part **b.**, $\lim_{x \rightarrow \infty} \sin \frac{1}{x} = 0$.
- As $x \rightarrow \infty$ both $\sin x$ and $\cos x$ oscillate between -1 and 1 .
- $$\lim_{x \rightarrow \infty} \left[\sin \left(x + \frac{1}{x} \right) - \sin x \right] = 0.$$
- 56.** $\lim_{v \rightarrow c^-} m(v) = \lim_{v \rightarrow c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$
- 57.** $\lim_{x \rightarrow \infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$
- 58.** $\lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$
- 59.** $\lim_{x \rightarrow -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5} \right) = -\frac{3}{2\sqrt{2}}$
- 60.** $\lim_{x \rightarrow \infty} \frac{2x+1}{\sqrt{3x^2+1}} = \frac{2}{\sqrt{3}}$
- 61.** $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^{10} = 1$
- 62.** $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e \approx 2.718$
- 63.** $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^{x^2} = \infty$
- 64.** $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^{\sin x} = 1$
- 65.** $\lim_{x \rightarrow 3^-} \frac{\sin|x-3|}{x-3} = -1$
- 66.** $\lim_{x \rightarrow 3^-} \frac{\sin|x-3|}{\tan(x-3)} = -1$
- 67.** $\lim_{x \rightarrow 3^-} \frac{\cos(x-3)}{x-3} = -\infty$
- 68.** $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{x - \frac{\pi}{2}} = -1$
- 69.** $\lim_{x \rightarrow 0^+} (1 + \sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$
- 70.** $\lim_{x \rightarrow 0^+} (1 + \sqrt{x})^{1/x} = \infty$
- 71.** $\lim_{x \rightarrow 0^+} (1 + \sqrt{x})^x = 1$

1.6 Concepts Review

- $\lim_{x \rightarrow c} f(x)$
- every integer
- $\lim_{x \rightarrow a^+} f(x) = f(a)$; $\lim_{x \rightarrow b^-} f(x) = f(b)$
- a ; b ; $f(c) = W$

Problem Set 1.6

- $\lim_{x \rightarrow 3} [(x-3)(x-4)] = 0 = f(3)$; continuous
- $\lim_{x \rightarrow 3} (x^2 - 9) = 0 = g(3)$; continuous
- $\lim_{x \rightarrow 3} \frac{3}{x-3}$ and $h(3)$ do not exist, so $h(x)$ is not continuous at 3.
- $\lim_{t \rightarrow 3} \sqrt{t-4}$ and $g(3)$ do not exist, so $g(t)$ is not continuous at 3.
- $\lim_{t \rightarrow 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist, so $h(t)$ is not continuous at 3.
- $h(3)$ does not exist, so $h(t)$ is not continuous at 3.
- $\lim_{t \rightarrow 3} |t| = 3 = f(3)$; continuous
- $\lim_{t \rightarrow 3} |t-2| = 1 = g(3)$; continuous
- $h(3)$ does not exist, so $h(t)$ is not continuous at 3.
- $f(3)$ does not exist, so $f(x)$ is not continuous at 3.
- $\lim_{t \rightarrow 3} \frac{t^3 - 27}{t-3} = \lim_{t \rightarrow 3} \frac{(t-3)(t^2 + 3t + 9)}{t-3}$
 $= \lim_{t \rightarrow 3} (t^2 + 3t + 9) = (3)^2 + 3(3) + 9 = 27 = r(3)$
 continuous
- From Problem 11, $\lim_{t \rightarrow 3} r(t) = 27$, so $r(t)$ is not continuous at 3 because $\lim_{t \rightarrow 3} r(t) \neq r(3)$.

- $\lim_{t \rightarrow 3^+} f(t) = \lim_{t \rightarrow 3^+} (3-t) = 0$
 $\lim_{t \rightarrow 3^-} f(t) = \lim_{t \rightarrow 3^-} (t-3) = 0$
 $\lim_{t \rightarrow 3} f(t) = f(3)$; continuous
- $\lim_{t \rightarrow 3^+} f(t) = \lim_{t \rightarrow 3^+} (3-t)^2 = 0$
 $\lim_{t \rightarrow 3^-} f(t) = \lim_{t \rightarrow 3^-} (t^2 - 9) = 0$
 $\lim_{t \rightarrow 3} f(t) = f(3)$; continuous
- $\lim_{t \rightarrow 3} f(x) = -2 = f(3)$; continuous
- g is discontinuous at $x = -3, 4, 6, 8$; g is left continuous at $x = 4, 8$; g is right continuous at $x = -3, 6$
- h is continuous on the intervals $(-\infty, -5)$, $[-5, 4]$, $(4, 6)$, $[6, 8]$, $(8, \infty)$
- $\lim_{x \rightarrow 7} \frac{x^2 - 49}{x-7} = \lim_{x \rightarrow 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x \rightarrow 7} (x+7) = 7 + 7 = 14$
 Define $f(7) = 14$.
- $\lim_{x \rightarrow 3} \frac{2x^2 - 18}{3-x} = \lim_{x \rightarrow 3} \frac{2(x+3)(x-3)}{3-x}$
 $= \lim_{x \rightarrow 3} [-2(x+3)] = -2(3+3) = -12$
 Define $f(3) = -12$.
- $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$
 Define $g(0) = 1$
- $\lim_{t \rightarrow 1} \frac{\sqrt{t}-1}{t-1} = \lim_{t \rightarrow 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}$
 $= \lim_{t \rightarrow 1} \frac{t-1}{(t-1)(\sqrt{t}+1)} = \lim_{t \rightarrow 1} \frac{1}{\sqrt{t}+1} = \frac{1}{2}$
 Define $H(1) = \frac{1}{2}$.
- $\lim_{x \rightarrow -1} \frac{x^4 + 2x^2 - 3}{x+1} = \lim_{x \rightarrow -1} \frac{(x^2-1)(x^2+3)}{x+1}$
 $= \lim_{x \rightarrow -1} \frac{(x+1)(x-1)(x^2+3)}{x+1}$
 $= \lim_{x \rightarrow -1} [(x-1)(x^2+3)]$
 $= (-1-1)[(-1)^2 + 3] = -8$
 Define $\phi(-1) = -8$.

23. $\lim_{x \rightarrow -1} \sin\left(\frac{x^2 - 1}{x + 1}\right) = \lim_{x \rightarrow -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$
 $= \lim_{x \rightarrow -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$
 Define $F(-1) = -\sin 2$.

24. Discontinuous at $x = \pi, 30$

25. $f(x) = \frac{33 - x^2}{(\pi - x)(x - 3)}$
 Discontinuous at $x = 3, \pi$

26. Continuous at all points

27. Discontinuous at all $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

28. Discontinuous at all $u \leq -5$

29. Discontinuous at $u = -1$

30. Continuous at all points

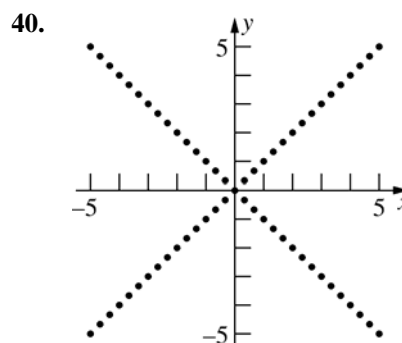
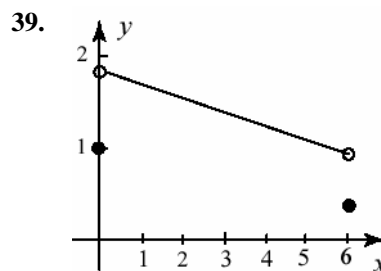
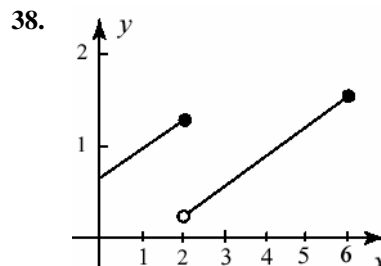
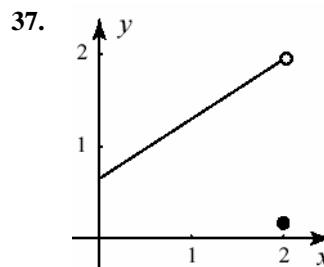
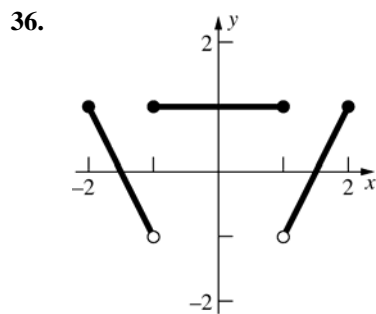
31. $G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$
 Discontinuous on $(-\infty, -2] \cup [2, \infty)$

32. Continuous at all points since
 $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ and $\lim_{x \rightarrow 1} f(x) = 1 = f(1)$.

33. $\lim_{x \rightarrow 0} g(x) = 0 = g(0)$
 $\lim_{x \rightarrow 1^+} g(x) = 1, \lim_{x \rightarrow 1^-} g(x) = -1$
 $\lim_{x \rightarrow 1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = 1$.

34. Discontinuous at every integer

35. Discontinuous at $t = n + \frac{1}{2}$ where n is any integer



Discontinuous at all points except $x = 0$, because $\lim_{x \rightarrow c} f(x) \neq f(c)$ for $c \neq 0$. $\lim_{x \rightarrow c} f(x)$ exists only at $c = 0$ and $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$.

41. Continuous.

42. Discontinuous: removable, define $f(10) = 20$

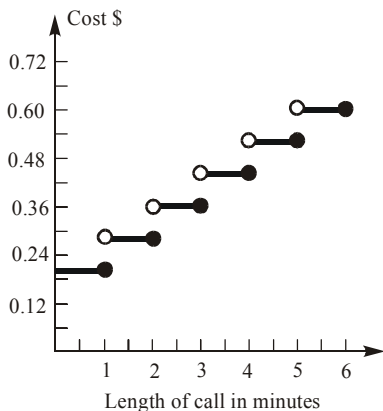
43. Discontinuous: removable, define $f(0) = 1$

44. Discontinuous: nonremovable.

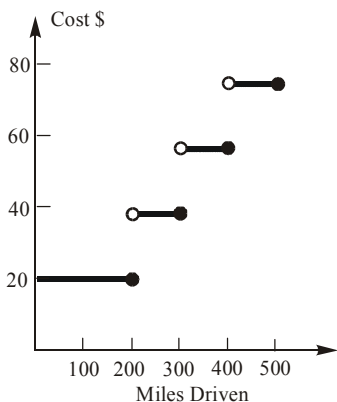
45. Discontinuous, removable, redefine $g(0) = 1$

46. Discontinuous: removable, define $F(0) = 0$

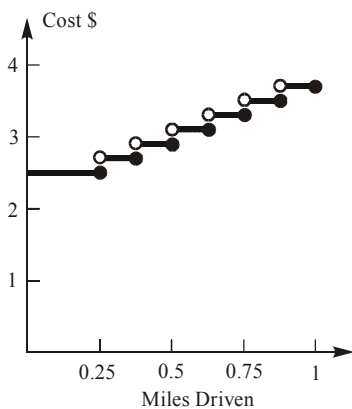
47. Discontinuous: nonremovable.
48. Discontinuous: removable, define $f(4) = 4$
49. The function is continuous on the intervals $(0, 1], (1, 2], (2, 3], \dots$



50. The function is continuous on the intervals $[0, 200], (200, 300], (300, 400], \dots$



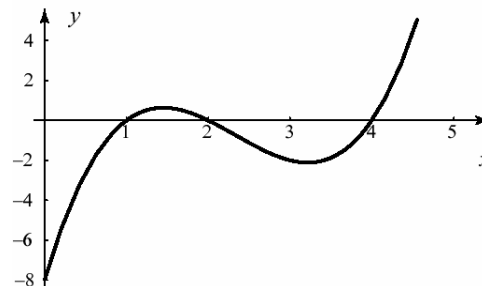
51. The function is continuous on the intervals $(0, 0.25], (0.25, 0.375], (0.375, 0.5], \dots$



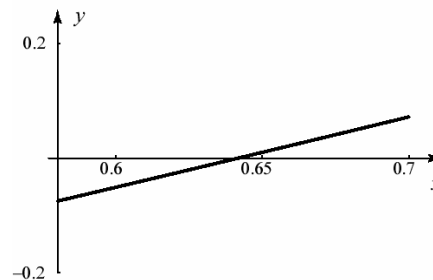
52. Let $f(x) = x^3 + 3x - 2$. f is continuous on $[0, 1]$. $f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

53. Because the function is continuous on $[0, 2\pi]$ and $(\cos 0)0^3 + 6\sin^5 0 - 3 = -3 < 0$, $(\cos 2\pi)(2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0$, there is at least one number c between 0 and 2π such that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$.

54. Let $f(x) = x^3 - 7x^2 + 14x - 8$. $f(x)$ is continuous at all values of x . $f(0) = -8, f(5) = 12$. Because 0 is between -8 and 12 , there is at least one number c between 0 and 5 such that $f(x) = x^3 - 7x^2 + 14x - 8 = 0$. This equation has three solutions ($x = 1, 2, 4$)



55. Let $f(x) = \sqrt{x} - \cos x$. $f(x)$ is continuous at all values of $x \geq 0$. $f(0) = -1, f(\pi/2) = \sqrt{\pi/2}$. Because 0 is between -1 and $\sqrt{\pi/2}$, there is at least one number c between 0 and $\pi/2$ such that $f(x) = \sqrt{x} - \cos x = 0$. The interval $[0.6, 0.7]$ contains the solution.



56. Let $f(x) = x^5 + 4x^3 - 7x + 14$. $f(x)$ is continuous at all values of x . $f(-2) = -36, f(0) = 14$. Because 0 is between -36 and 14 , there is at least one number c between -2 and 0 such that $f(x) = x^5 + 4x^3 - 7x + 14 = 0$.

57. Suppose that f is continuous at c , so
 $\lim_{x \rightarrow c} f(x) = f(c)$. Let $x = c + t$, so $t = x - c$, then
as $x \rightarrow c$, $t \rightarrow 0$ and the statement
 $\lim_{x \rightarrow c} f(x) = f(c)$ becomes $\lim_{t \rightarrow 0} f(t + c) = f(c)$.
Suppose that $\lim_{t \rightarrow 0} f(t + c) = f(c)$ and let $x = t + c$,
so $t = x - c$. Since c is fixed, $t \rightarrow 0$ means that
 $x \rightarrow c$ and the statement $\lim_{t \rightarrow 0} f(t + c) = f(c)$
becomes $\lim_{x \rightarrow c} f(x) = f(c)$, so f is continuous at
 c .
58. Since $f(x)$ is continuous at c ,
 $\lim_{x \rightarrow c} f(x) = f(c) > 0$. Choose $\varepsilon = f(c)$, then
there exists a $\delta > 0$ such that
 $0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
Thus, $f(x) - f(c) > -\varepsilon = -f(c)$, or $f(x) > 0$.
Since also $f(c) > 0$, $f(x) > 0$ for all x in
 $(c - \delta, c + \delta)$.
59. Let $g(x) = x - f(x)$. Then,
 $g(0) = 0 - f(0) = -f(0) \leq 0$ and $g(1) = 1 - f(1) \geq 0$
since $0 \leq f(x) \leq 1$ on $[0, 1]$. If $g(0) = 0$, then
 $f(0) = 0$ and $c = 0$ is a fixed point of f . If $g(1) = 0$,
then $f(1) = 1$ and $c = 1$ is a fixed point of f . If
neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and
 $g(1) > 0$ so there is some c in $[0, 1]$ such that
 $g(c) = 0$. If $g(c) = 0$ then $c - f(c) = 0$ or
 $f(c) = c$ and c is a fixed point of f .
60. For $f(x)$ to be continuous everywhere,
 $f(1) = a(1) + b = 2$ and $f(2) = 6 = a(2) + b$
 $a + b = 2$
 $2a + b = 6$
 $-a = -4$
 $a = 4, b = -2$
61. For x in $[0, 1]$, let $f(x)$ indicate where the string
originally at x ends up. Thus $f(0) = a, f(1) = b$.
 $f(x)$ is continuous since the string is unbroken.
Since $0 \leq a, b \leq 1$, $f(x)$ satisfies the conditions of
Problem 59, so there is some c in $[0, 1]$ with
 $f(c) = c$, i.e., the point of string originally at c
ends up at c .
62. The Intermediate Value Theorem does not imply
the existence of a number c between -2 and 2
such that $f(c) = 0$. The reason is that the
function $f(x)$ is not continuous on $[-2, 2]$.
63. Let $f(x)$ be the difference in times on the hiker's
watch where x is a point on the path, and suppose
 $x = 0$ at the bottom and $x = 1$ at the top of the
mountain.
So $f(x) = (\text{time on watch on the way up}) - (\text{time on watch on the way down})$.
 $f(0) = 4 - 11 = -7, f(1) = 12 - 5 = 7$. Since time is
continuous, $f(x)$ is continuous, hence there is
some c between 0 and 1 where $f(c) = 0$. This c is
the point where the hiker's watch showed the
same time on both days.
64. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is
the length of the side of the rectangle which
makes angle θ with the x -axis minus the length of
the sides perpendicular to it. f is continuous on
 $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is
circumscribed by a square. If $f(0) \neq 0$, then
observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the
Intermediate Value Theorem, there is an angle
 θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$.
Hence, D can be circumscribed by a square.
65. Yes, g is continuous at R .

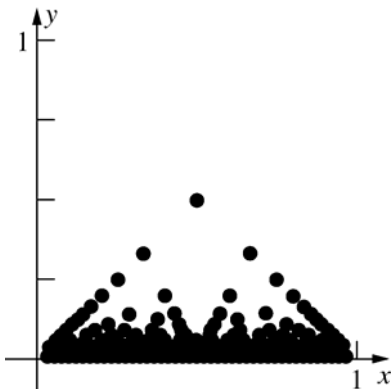
$$\lim_{r \rightarrow R^-} g(r) = \frac{GMm}{R^2} = \lim_{r \rightarrow R^+} g(r)$$
66. No. By the Intermediate Value Theorem, if f
were to change signs on $[a, b]$, then f must be
 0 at some c in $[a, b]$. Therefore, f cannot
change sign.
67. a. $f(x) = f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. We
want to prove that $\lim_{x \rightarrow c} f(x) = f(c)$, or,
equivalently, $\lim_{x \rightarrow c} [f(x) - f(c)] = 0$. But
 $f(x) - f(c) = f(x - c)$, so
 $\lim_{x \rightarrow c} [f(x) - f(c)] = \lim_{x \rightarrow c} f(x - c)$. Let
 $h = x - c$ then as $x \rightarrow c$, $h \rightarrow 0$ and
 $\lim_{x \rightarrow c} f(x - c) = \lim_{h \rightarrow 0} f(h) = f(0) = 0$. Hence
 $\lim_{x \rightarrow c} f(x) = f(c)$ and f is continuous at c .
Thus, f is continuous everywhere, since c
was arbitrary.
- b. By Problem 43 of Section 0.5, $f(t) = mt$ for
all t in \mathbf{Q} . Since $g(t) = mt$ is a polynomial
function, it is continuous for all real
numbers. $f(t) = g(t)$ for all t in \mathbf{Q} , thus
 $f(t) = g(t)$ for all t in \mathbf{R} , i.e. $f(t) = mt$.

68. If $f(x)$ is continuous on an interval then $\lim_{x \rightarrow c} f(x) = f(c)$ for all points in the interval:

$$\begin{aligned} \lim_{x \rightarrow c} f(x) = f(c) &\Rightarrow \lim_{x \rightarrow c} |f(x)| \\ &= \lim_{x \rightarrow c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x \rightarrow c} f(x)\right)^2} \\ &= \sqrt{(f(c))^2} = |f(c)| \end{aligned}$$

69. Suppose $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

70. a.



- b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left| \frac{1}{q} - 0 \right| = \frac{1}{q}$. Hence, $\lim_{x \rightarrow r} f(x)$ does not exist. If c is any irrational number in $(0, 1)$, then as $x = \frac{p}{q} \rightarrow c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \rightarrow \infty$, so $f(x) \rightarrow 0$ as $x \rightarrow c$. Thus, $\lim_{x \rightarrow c} f(x) = 0 = f(c)$ for any irrational number c .

71. a. Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

$$\text{Domain: } \left[-\frac{3}{4}, \frac{3}{4}\right];$$

$$\text{Range: } \left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$$

- b. At $x = 0$
- c. If $x = 0$, $f(x) = 0$, if $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, 0, \frac{3}{4}$ are fixed points of f .

1.7 Chapter Review

Concepts Test

- False. Consider $f(x) = \lceil x \rceil$ at $x = 2$.
- False: c may not be in the domain of $f(x)$, or it may be defined separately.
- False: c may not be in the domain of $f(x)$, or it may be defined separately.
- True. By definition, where $c = 0, L = 0$.
- False: If $f(c)$ is not defined, $\lim_{x \rightarrow c} f(x)$ might exist; e.g., $f(x) = \frac{x^2 - 4}{x + 2}$. $f(-2)$ does not exist, but $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = -4$.
- True: $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = \lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)}{x - 5} = \lim_{x \rightarrow 5} (x + 5) = 5 + 5 = 10$
- True: Substitution Theorem
- False: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- False: The tangent function is not defined for all values of c .
- True: If x is in the domain of $\tan x = \frac{\sin x}{\cos x}$, then $\cos x \neq 0$, and Theorem A.7 applies..

11. True: Since both $\sin x$ and $\cos x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x) = 2 \sin^2 x - \cos x$ is also continuous for all real numbers.
12. True. By definition, $\lim_{x \rightarrow c} f(x) = f(c)$.
13. True. $2 \in [1, 3]$
14. False: $\lim_{x \rightarrow 0^-}$ may not exist
15. False: Consider $f(x) = \sin x$.
16. True. By the definition of continuity on an interval.
17. False: Since $-1 \leq \sin x \leq 1$ for all x and $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, we get $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.
18. False. It could be the case where $\lim_{x \rightarrow -\infty} f(x) = 2$
19. False: The graph has many vertical asymptotes; e.g., $x = \pm \pi/2, \pm 3\pi/2, \dots$
20. True: $x = 2$; $x = -2$
21. True: As $x \rightarrow 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.
22. False: $\lim_{x \rightarrow c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.
23. True: $\lim_{x \rightarrow c} f(x) = f\left(\lim_{x \rightarrow c} x\right) = f(c)$, so f is continuous at $x = c$.
24. True: $\lim_{x \rightarrow 2.3} \left\lfloor \frac{x}{2} \right\rfloor = 1 = f(2.3)$
25. True: Choose $\varepsilon = 0.001f(2)$ then since $\lim_{x \rightarrow 2} f(x) = f(2)$, there is some δ such that $0 < |x - 2| < \delta \Rightarrow |f(x) - f(2)| < 0.001f(2)$, or $-0.001f(2) < f(x) - f(2) < 0.001f(2)$. Thus, $0.999f(2) < f(x) < 1.001f(2)$ and $f(x) < 1.001f(2)$ for $0 < |x - 2| < \delta$. Since $f(2) < 1.001f(2)$, as $f(2) > 0$, $f(x) < 1.001f(2)$ on $(2 - \delta, 2 + \delta)$.
26. False: That $\lim_{x \rightarrow c} [f(x) + g(x)]$ exists does not imply that $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and $g(x) = \frac{x+7}{x+2}$ for $c = -2$.
27. True: Squeeze Theorem
28. True: A function has only one limit at a point, so if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} f(x) = M$, $L = M$
29. False: That $f(x) \neq g(x)$ for all x does not imply that $\lim_{x \rightarrow c} f(x) \neq \lim_{x \rightarrow c} g(x)$. For example, if $f(x) = \frac{x^2 + x - 6}{x - 2}$ and $g(x) = \frac{5}{2}x$, then $f(x) \neq g(x)$ for all x , but $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 5$.
30. False: If $f(x) < 10$, $\lim_{x \rightarrow 2} f(x)$ could equal 10 if there is a discontinuity point $(2, 10)$. For example, $f(x) = \frac{-x^3 + 6x^2 - 2x - 12}{x - 2} < 10$ for all x , but $\lim_{x \rightarrow 2} f(x) = 10$.
31. True: $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} \sqrt{f^2(x)} = \sqrt{\left[\lim_{x \rightarrow a} f(x)\right]^2} = \sqrt{(b)^2} = |b|$
32. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

- $\lim_{x \rightarrow 2} \frac{x-2}{x+2} = \frac{2-2}{2+2} = \frac{0}{4} = 0$
- $\lim_{u \rightarrow 1} \frac{u^2-1}{u+1} = \frac{1^2-1}{1+1} = 0$
- $\lim_{u \rightarrow 1} \frac{u^2-1}{u-1} = \lim_{u \rightarrow 1} \frac{(u-1)(u+1)}{u-1} = \lim_{u \rightarrow 1} (u+1) = 1+1 = 2$
- $\lim_{u \rightarrow 1} \frac{u+1}{u^2-1} = \lim_{u \rightarrow 1} \frac{u+1}{(u+1)(u-1)} = \lim_{u \rightarrow 1} \frac{1}{u-1}$;
does not exist
- $\lim_{x \rightarrow 2} \frac{1-\frac{2}{x}}{x^2-4} = \lim_{x \rightarrow 2} \frac{\frac{x-2}{x}}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x(x+2)} = \frac{1}{2(2+2)} = \frac{1}{8}$
- $\lim_{z \rightarrow 2} \frac{z^2-4}{z^2+z-6} = \lim_{z \rightarrow 2} \frac{(z+2)(z-2)}{(z+3)(z-2)} = \lim_{z \rightarrow 2} \frac{z+2}{z+3} = \frac{2+2}{2+3} = \frac{4}{5}$
- $\lim_{x \rightarrow 0} \frac{\tan x}{\sin 2x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{2 \sin x \cos x} = \lim_{x \rightarrow 0} \frac{1}{2 \cos^2 x} = \frac{1}{2 \cos^2 0} = \frac{1}{2}$
- $\lim_{y \rightarrow 1} \frac{y^3-1}{y^2-1} = \lim_{y \rightarrow 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)} = \lim_{y \rightarrow 1} \frac{y^2+y+1}{y+1} = \frac{1^2+1+1}{1+1} = \frac{3}{2}$
- $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \rightarrow 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim_{x \rightarrow 4} (\sqrt{x}+2) = \sqrt{4}+2 = 4$
- $\lim_{x \rightarrow 0} \frac{\cos x}{x}$ does not exist.
- $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1$
- $\lim_{x \rightarrow (1/2)^+} \lceil 4x \rceil = 2$
- $\lim_{t \rightarrow 2^-} (\lceil t \rceil - t) = \lim_{t \rightarrow 2^-} \lceil t \rceil - \lim_{t \rightarrow 2^-} t = 1 - 2 = -1$
- $\lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} = \lim_{x \rightarrow 1^-} \frac{1-x}{x-1} = -1$ since $x-1 < 0$ as $x \rightarrow 1^-$
- $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \frac{5 \sin 5x}{3 \cdot 5x} = \frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = \frac{5}{3} \times 1 = \frac{5}{3}$
- $\lim_{x \rightarrow 0} \frac{1-\cos 2x}{3x} = \lim_{x \rightarrow 0} \frac{2}{3} \frac{1-\cos 2x}{2x} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{1-\cos 2x}{2x} = \frac{2}{3} \times 0 = 0$
- $\lim_{x \rightarrow \infty} \frac{x-1}{x+2} = \lim_{x \rightarrow \infty} \frac{1-\frac{1}{x}}{1+\frac{2}{x}} = \frac{1+0}{1+0} = 1$
- Since $-1 \leq \sin t \leq 1$ for all t and $\lim_{t \rightarrow \infty} \frac{1}{t} = 0$, we get $\lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0$.
- $\lim_{t \rightarrow 2} \frac{t+2}{(t-2)^2} = \infty$ because as $t \rightarrow 0$, $t+2 \rightarrow 4$ while the denominator goes to 0 from the right.
- $\lim_{x \rightarrow 0^+} \frac{\cos x}{x} = \infty$, because as $x \rightarrow 0^+$, $\cos x \rightarrow 1$ while the denominator goes to 0 from the right.
- $\lim_{x \rightarrow \pi/4^-} \tan 2x = \infty$ because as $x \rightarrow (\pi/4)^-$, $2x \rightarrow (\pi/2)^-$, so $\tan 2x \rightarrow \infty$.
- $\lim_{x \rightarrow 0^+} \frac{1+\sin x}{x} = \infty$, because as $x \rightarrow 0^+$, $1+\sin x \rightarrow 1$ while the denominator goes to 0 from the right.
- Preliminary analysis: Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that
 $0 < |x-3| < \delta \Rightarrow |(2x+1)-7| < \varepsilon$
 $|2x-6| < \varepsilon \Leftrightarrow 2|x-3| < \varepsilon$
 $\Leftrightarrow |x-3| < \frac{\varepsilon}{2}$. Choose $\delta = \frac{\varepsilon}{2}$.
 Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$. Thus,
 $|(2x+1)-7| = |2x-6| = 2|x-3| < 2(\varepsilon/2) = \varepsilon$.

24. a. $f(1) = 0$

b. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1 - x) = 0$

c. $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1$

d. $\lim_{x \rightarrow -1} f(x) = -1$ because
 $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^3 = -1$ and
 $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} x = -1$

25. a. f is discontinuous at $x = 1$ because $f(1) = 0$, but $\lim_{x \rightarrow 1} f(x)$ does not exist. f is discontinuous at $x = -1$ because $f(-1)$ does not exist.

b. Define $f(-1) = -1$

26. a. $0 < |u - a| < \delta \Rightarrow |g(u) - M| < \varepsilon$

b. $0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon$

27. a. $\lim_{x \rightarrow 3} [2f(x) - 4g(x)]$
 $= 2 \lim_{x \rightarrow 3} f(x) - 4 \lim_{x \rightarrow 3} g(x)$
 $= 2(3) - 4(-2) = 14$

b. $\lim_{x \rightarrow 3} g(x) \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} g(x)(x + 3)$
 $= \lim_{x \rightarrow 3} g(x) \cdot \lim_{x \rightarrow 3} (x + 3) = -2 \cdot (3 + 3) = -12$

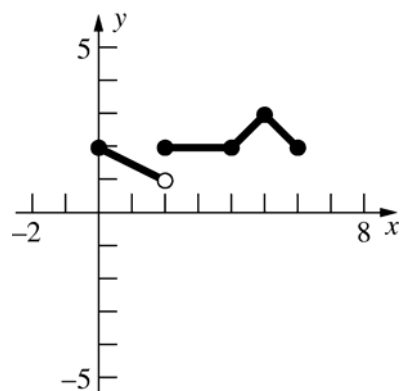
c. $g(3) = -2$

d. $\lim_{x \rightarrow 3} g(f(x)) = g\left(\lim_{x \rightarrow 3} f(x)\right) = g(3) = -2$

e. $\lim_{x \rightarrow 3} \sqrt{f^2(x) - 8g(x)}$
 $= \sqrt{\left[\lim_{x \rightarrow 3} f(x)\right]^2 - 8 \lim_{x \rightarrow 3} g(x)}$
 $= \sqrt{(3)^2 - 8(-2)} = 5$

f. $\lim_{x \rightarrow 3} \frac{|g(x) - g(3)|}{f(x)} = \frac{|-2 - g(3)|}{3} = \frac{|-2 - (-2)|}{3} = 0$

28.



29. $a(0) + b = -1$ and $a(1) + b = 1$
 $b = -1$; $a + b = 1$
 $a - 1 = 1$
 $a = 2$

30. Let $f(x) = x^5 - 4x^3 - 3x + 1$
 $f(2) = -5$, $f(3) = 127$
 Because $f(x)$ is continuous on $[2, 3]$ and $f(2) < 0 < f(3)$, there exists some number c between 2 and 3 such that $f(c) = 0$.

31. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x \rightarrow \infty} \frac{x}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{x}{x^2 + 1} = 0$, so $y = 0$ is a horizontal asymptote.

32. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 1} = 1$, so $y = 1$ is a horizontal asymptote.

33. Vertical: $x = 1, x = -1$ because $\lim_{x \rightarrow 1^+} \frac{x^2}{x^2 - 1} = \infty$

and $\lim_{x \rightarrow -1^-} \frac{x^2}{x^2 - 1} = \infty$

Horizontal: $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2 - 1} = 1$, so $y = 1$ is a horizontal asymptote.

34. Vertical: $x = 2, x = -2$ because

$$\lim_{x \rightarrow 2^+} \frac{x^3}{x^2 - 4} = \infty \text{ and } \lim_{x \rightarrow -2^-} \frac{x^3}{x^2 - 4} = \infty$$

Horizontal: $\lim_{x \rightarrow \infty} \frac{x^3}{x^2 - 4} = \infty$ and

$\lim_{x \rightarrow -\infty} \frac{x^3}{x^2 - 4} = -\infty$, so there are no horizontal asymptotes.

35. Vertical: $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \dots$ because

$\lim_{x \rightarrow \pi/4^-} \tan 2x = \infty$ and similarly for other odd multiples of $\pi/4$.

Horizontal: None, because $\lim_{x \rightarrow \infty} \tan 2x$ and

$\lim_{x \rightarrow -\infty} \tan 2x$ do not exist.

36. Vertical: $x = 0$, because

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x} \frac{\sin x}{x} = \infty.$$

Horizontal: $y = 0$, because

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x^2} = \lim_{x \rightarrow -\infty} \frac{\sin x}{x^2} = 0.$$

2. a. $g(2) = 1/2$

b. $g(2.1) = 1/2.1 \approx 0.476$

c. $g(2.1) - g(2) = 0.476 - 0.5 = -0.024$

d. $\frac{g(2.1) - g(2)}{2.1 - 2} = \frac{-0.024}{0.1} = -0.24$

e. $g(a+h) = 1/(a+h)$

f. $g(a+h) - g(a) = 1/(a+h) - 1/a = \frac{-h}{a(a+h)}$

g. $\frac{g(a+h) - g(a)}{(a+h) - a} = \frac{\frac{-h}{a(a+h)}}{h} = \frac{-1}{a(a+h)}$

h. $\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{(a+h) - a} = \frac{-1}{a^2}$

3. a. $F(2) = \sqrt{2} \approx 1.414$

b. $F(2.1) = \sqrt{2.1} \approx 1.449$

c. $F(2.1) - F(2) = 1.449 - 1.414 = 0.035$

d. $\frac{F(2.1) - F(2)}{2.1 - 2} = \frac{0.035}{0.1} = 0.35$

e. $F(a+h) = \sqrt{a+h}$

f. $F(a+h) - F(a) = \sqrt{a+h} - \sqrt{a}$

g. $\frac{F(a+h) - F(a)}{(a+h) - a} = \frac{\sqrt{a+h} - \sqrt{a}}{h}$

h. $\lim_{h \rightarrow 0} \frac{F(a+h) - F(a)}{(a+h) - a} = \lim_{h \rightarrow 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{a+h} - \sqrt{a})(\sqrt{a+h} + \sqrt{a})}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \rightarrow 0} \frac{a+h-a}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = \frac{\sqrt{a}}{2a}$$

Review and Preview Problems

1. a. $f(2) = 2^2 = 4$

b. $f(2.1) = 2.1^2 = 4.41$

c. $f(2.1) - f(2) = 4.41 - 4 = 0.41$

d. $\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{0.41}{0.1} = 4.1$

e. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$

f. $f(a+h) - f(a) = a^2 + 2ah + h^2 - a^2 = 2ah + h^2$

g. $\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{2ah + h^2}{h} = 2a + h$

h. $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \rightarrow 0} (2a + h) = 2a$

4. a. $G(2) = (2)^3 + 1 = 8 + 1 = 9$
- b. $G(2.1) = (2.1)^3 + 1 = 9.261 + 1 = 10.261$
- c. $G(2.1) - G(2) = 10.261 - 9 = 1.261$
- d. $\frac{G(2.1) - G(2)}{2.1 - 2} = \frac{1.261}{0.1} = 12.61$
- e. $G(a+h) = (a+h)^3 + 1$
 $= a^3 + 3a^2h + 3ah^2 + h^3 + 1$
- f. $G(a+h) - G(a) = [(a+h)^3 + 1] - [a^3 + 1]$
 $= (a^3 + 3a^2h + 3ah^2 + h^3 + 1) - (a^3 + 1)$
 $= 3a^2h + 3ah^2 + h^3$
- g. $\frac{G(a+h) - G(a)}{(a+h) - a} = \frac{3a^2h + 3ah^2 + h^3}{h}$
 $= 3a^2 + 3ah + h^2$
- h. $\lim_{h \rightarrow 0} \frac{G(a+h) - G(a)}{(a+h) - a} = \lim_{h \rightarrow 0} 3a^2 + 3ah + h^2$
 $= 3a^2$
5. a. $(a+b)^3 = a^3 + 3a^2b + \dots$
- b. $(a+b)^4 = a^4 + 4a^3b + \dots$
- c. $(a+b)^5 = a^5 + 5a^4b + \dots$
6. $(a+b)^n = a^n + na^{n-1}b + \dots$
7. $\sin(x+h) = \sin x \cos h + \cos x \sin h$
8. $\cos(x+h) = \cos x \cos h - \sin x \sin h$
9. a. The point will be at position (10,0) in all three cases ($t = 1, 2, 3$) because it will have made 4, 8, and 12 revolutions respectively.
- b. Since the point is rotating at a rate of 4 revolutions per second, it will complete 1 revolution after $\frac{1}{4}$ second. Therefore, the point will first return to its starting position at time $t = \frac{1}{4}$.
10. $V_0 = \frac{4}{3}\pi(2)^3 = \frac{32\pi}{3} \text{ cm}^3$
 $V_1 = \frac{4}{3}\pi(2.5)^3 = \frac{62.5\pi}{3} = \frac{125\pi}{6} \text{ cm}^3$
 $\Delta V = V_1 - V_0 = \frac{125\pi}{6} \text{ cm}^3 - \frac{32\pi}{3} \text{ cm}^3$
 $= \frac{61}{6}\pi \text{ cm}^3 \approx 31.940 \text{ cm}^3$
11. a. North plane has traveled 600 miles. East plane has traveled 400 miles.
- b. $d = \sqrt{600^2 + 400^2}$
 $= 721 \text{ miles}$
- c. $d = \sqrt{675^2 + 500^2}$
 $= 840 \text{ miles}$