CHAPTER TWO

Student's Solutions Manual

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FOR

A Course In

Probability

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Preface

This *Student's Solutions Manual* (SSM) is designed to be used with the book *A Course in Probability* (CIP) by Neil A. Weiss. The SSM provides complete and detailed solutions to every fourth end-of-section exercise and to all review exercises. Thus, for end-of-section exercises, you will find solutions to problems numbered 1, 5, 9, . . . , whereas, solutions to all review problems are presented.

The solutions in this SSM employ precisely the same notation, format, and style as those given to the examples in CIP. Consequently, you need only concentrate on the exercise solutions themselves without having to struggle with new notations or conventions.

We would like to express our appreciation to all the people at Addison-Wesley who helped make this SSM possible. In particular, we thank Deirdre Lynch, Sara Oliver Gordus, Christina Lepre, Kayla Smith-Tarbox, and Joe Vetere. And, in addition, we thank Carol Weiss for her outstanding job of composition and proofreading.

Ann Arbor, Michigan *A.A.*
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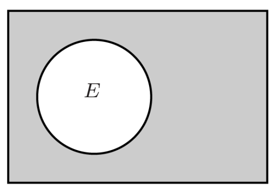
Chapter 2

Mathematical Probability

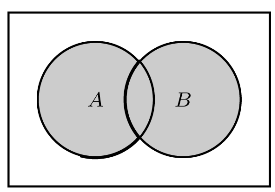
2.1 Sample Space and Events

Basic Exercises

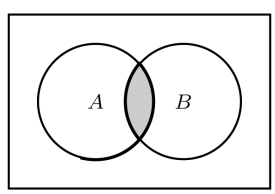
2.1 In each case, the shaded region represents the event in question.
a) E^c is the event that E does not occur:



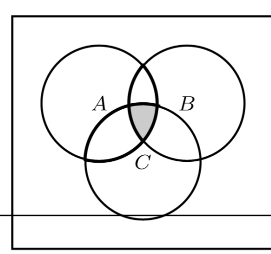
b) $A \cup B$ is the event that either A or B or both occur (i.e., at least one of A and B occurs):



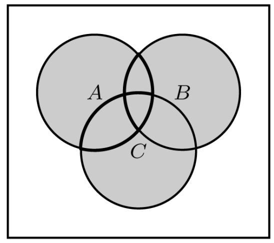
c) $A \cap B$ is the event that both A and B occur:



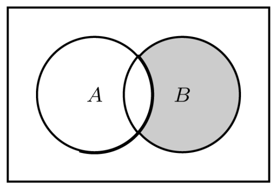
d) $A \cap B \cap C$ is the event that all three of A, B, and C occur:



e) $A \cup B \cup C$ is the event that at least one of A, B, and C occurs:



f) $A^c \cap B$ is the event that B occurs but A doesn't occur:



2.5

a) The outcome of the random experiment is the tree obtained. Hence, the sample space consists of the set of 39 trees.
b) Event B^c is that the tree obtained does not have at least 20% seed damage; in other words, that it has less than 20% seed damage. This event is comprised of $19 + 2 = 21$ trees.
c) Event $C \cap D$ is that the tree obtained has at least 30% but less than 60% seed damage and that it has at least 50% seed damage; in other words, that it has at least 50% but less than 60% seed damage. This event is comprised of two trees.
d) Event $A \cup D$ is that the tree obtained has either less than 40% seed damage or at least 50% seed damage. This event is comprised of $19 + 2 + 5 + 3 + 2 + 2 = 33$ trees.
e) Event C^c is that the tree obtained does not have at least 30% but less than 60% seed damage; in other words, that it has either less than 30% seed damage or at least 60% seed damage. This event is comprised of $19 + 2 + 5 + 2 = 28$ trees.
f) Event $A \cap D$ is that the tree obtained has less than 40% seed damage and at least 50% seed damage, which is impossible. This event is comprised of zero trees.
g) We note that $A \cap B \neq \emptyset$, $A \cap C \neq \emptyset$, $A \cap D = \emptyset$, $B \cap C \neq \emptyset$, $B \cap D \neq \emptyset$, and $C \cap D \neq \emptyset$. Therefore, events A and D are mutually exclusive, whereas no other two events among A, B, C, and D are mutually exclusive. Moreover, because any collection of three or four events among A, B, C, and D must contain two events that do not consist of A and D, we deduce that no such collection consists of mutually exclusive events.

2.9

a) A typical outcome of this random experiment can be represented as a string of heads followed by a single tail. Hence, a sample space for this random experiment is $\Omega = \{\text{T}, \text{HT}, \text{HHT}, \text{HHHT}, \dots\}$.
b) Let E denote the event that Laura wins. We note that E occurs if and only if the first tail occurs on an even-numbered toss. Hence, we have that $E = \{\text{HT}, \text{HHHT}, \text{HHHHHT}, \dots\}$.

2.13

a) Event A occurs but event B doesn't occur if and only if the outcome, ω, of the random experiment is a member of A but not of B, which means that $\omega \in A$ and $\omega \in B^c$; that is, $\omega \in A \cap B^c$. Thus, the event that A occurs but B doesn't occur is $A \cap B^c$.

b) Exactly one of A and B occurs if and only if either A occurs but B doesn't occur or B occurs but A doesn't occur. Referring to part (a), we conclude that the event that exactly one of A and B occurs is $(A \cap B^c) \cup (A^c \cup B)$.
c) We first note that, for events E, F, and G, event E occurs but events F and G don't occur if and only if event E occurs but event $F \cup G$ doesn't occur. From part (a), then, the event that E occurs but F and G don't occur is

$$E \cap (F \cup G)^c = E \cap \left(F^c \cap G^c\right) = E \cap F^c \cap G^c.$$

Now, exactly one of A, B, and C occurs if and only if A occurs but B and C don't occur or B occurs but A and C don't occur or C occurs but A and B don't occur. Therefore, the event that exactly one of A, B, and C occurs is $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$ or, equivalently,

$$\left(A \cap B^c \cap C^c\right) \cup \left(A^c \cap B \cap C^c\right) \cup \left(A^c \cap B^c \cap C\right).$$

d) At most two of the events A, B, and C occur if and only if not all three of the events occur, which is the event $(A \cap B \cap C)^c$ or, equivalently, $A^c \cup B^c \cup C^c$.

Advanced Exercises

2.17
a) If $\Omega = \{a, b\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, and Ω.
b) If $\Omega = \{a, b, c\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and Ω.
c) If $\Omega = \{a, b, c, d\}$, then the events are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and Ω.
d) From parts (a)–(c), we see that, if Ω has exactly two, three, or four elements, then it has 4, 8, and 16 events, respectively. We therefore claim that, if Ω has exactly n elements, then it has 2^n events. To prove that result, we use mathematical induction. From part (a), the result is true for $n = 2$. Assuming its truth for $n - 1$, we prove it for n. So, suppose that Ω has exactly n elements, say, $\Omega = \{a_1, \dots, a_n\}$. The number of events of Ω that do not contain a_n equals the total number of events of $\Omega' = \{a_1, \dots, a_{n-1}\}$, which, by the induction assumption is 2^{n-1}. By adding a_n to each of those 2^{n-1} events, we obtain all the events of Ω that do contain a_n. Hence, the total number of events of Ω is $2^{n-1} + 2^{n-1} = 2^n$, as required.

2.2 Axioms of Probability

Basic Exercises

2.21 As noted, here $\Omega = \{\text{H}, \text{T}\}$.
a) The numbers p and $1 - p$ are nonnegative and sum to 1. Hence, by Proposition 2.3 on page 43, there is a unique probability measure on the events of Ω such that $P(\{\text{H}\}) = p$ and $P(\{\text{T}\}) = 1 - p$.
b) The four events of this random experiment are $\emptyset$, $\{\text{H}\}$, $\{\text{T}\}$, and Ω. We know that $P(\{\text{H}\}) = p$ and $P(\{\text{T}\}) = 1 - p$. Moreover, for any probability measure, we have $P(\emptyset) = 0$ and $P(\Omega) = 1$.

2.25
a) We have

$$1/36 \geq 0 \qquad \text{and} \qquad \underbrace{1/36 + \cdots + 1/36}_{36 \text{ times}} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.

b) Let E be an event of this random experiment. Applying Proposition 2.2 on page 42 with the probability assignment from part (a), we get that

$$P(E) = \sum_{\omega \in E} P(\{\omega\}) = \sum_{\omega \in E} \frac{1}{36} = \frac{N(E)}{36},$$

where $N(E)$ denotes the number of outcomes that comprise event E.

By referring to the solution to Exercise 2.7, we then find that

$$\begin{aligned}
P(A_2) &= P(\{(1,1)\}) = \frac{1}{36} \\
P(A_3) &= P(\{(1,2),(2,1)\}) = \frac{2}{36} \\
P(A_4) &= P(\{(1,3),(2,2),(3,1)\}) = \frac{3}{36} \\
P(A_5) &= P(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36} \\
P(A_6) &= P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36} \\
P(A_7) &= P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36} \\
P(A_8) &= P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36} \\
P(A_9) &= P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36} \\
P(A_{10}) &= P(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36} \\
P(A_{11}) &= P(\{(5,6),(6,5)\}) = \frac{2}{36} \\
P(A_{12}) &= P(\{(6,6)\}) = \frac{1}{36}
\end{aligned}$$

More concisely, we have

$$P(A_i) = \begin{cases} (i-1)/36, & 2 \le i \le 7; \\ (13-i)/36, & 8 \le i \le 12. \end{cases}$$

c) Answers will vary. In view of Proposition 2.2, any assignment of 36 numbers to the 36 possible outcomes will work provided that those numbers are nonnegative and sum to 1. To obtain the probabilities of the A_is using your probability assignment, apply Proposition 2.2.

d) Assuming the die is balanced, the 36 possible outcomes are equally likely and, hence, the only proper assignment of probabilities is the one presented in part (a). In particular, then, the assignment in part (c), being different from the one in part (a), is not reasonable.

Theory Exercises

2.29
a) If $P(A_i) = 1/6$ for each i and $n = 10$, then Relation $(*)$ tells us that

$$P(A_1 \cup \cdots \cup A_{10}) \leq P(A_1) + \cdots + P(A_{10}) = \underbrace{1/6 + \cdots + 1/6}_{10 \text{ times}} = \frac{5}{3},$$

that is, that the union of the 10 events has probability at most 5/3. This result is trivial because the probability of any event must be at most 1.

If $P(A_i) = 1/60$ for each i and $n = 10$, then Relation $(*)$ tells us that

$$P(A_1 \cup \cdots \cup A_{10}) \leq P(A_1) + \cdots + P(A_{10}) = \underbrace{1/60 + \cdots + 1/60}_{10 \text{ times}} = \frac{1}{6},$$

that is, that the union of the 10 events has probability at most 1/6. This result certainly conveys some useful information. Likewise, if $P(A_i) = 1/6$ for each i and $n = 4$, then Relation $(*)$ tells us that

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) \leq P(A_1) + P(A_2) + P(A_3) + P(A_4) = 1/6 + 1/6 + 1/6 + 1/6 = \frac{2}{3},$$

that is, that the union of the four events has probability at most 2/3. This result also conveys some useful information.

b) We use mathematical induction to prove Relation $(*)$. For $n = 1$, it states that $P(A_1) \leq P(A_1)$, which is certainly true. Assuming its truth for $n - 1$, we prove it for n. For convenience, set $B_n = \bigcup_{i=1}^{n-1} A_i$. From the induction assumption, we have

$$P(B_n) = P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} P(A_i).$$

Now we apply, in turn, Exercises 2.24(b) and 2.24(a) with $A = B_n$ and $B = A_n$, and the nonnegativity axiom to deduce that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(B_n \cup A_n) = P(B_n) + P(A_n \cap B_n^c) = P(B_n) + P(A_n) - P(A_n \cap B_n)$$

$$\leq P(B_n) + P(A_n) \leq \sum_{i=1}^{n-1} P(A_i) + P(A_n) = \sum_{i=1}^{n} P(A_i),$$

as required.

Advanced Exercises

2.33 Here the sample space is the unit square: $\Omega = [0, 1] \times [0, 1]$. We have

$$P(\{\omega\}) = \text{Area}(\{\omega\}) = 0, \qquad \omega \in \Omega.$$

We claim that Ω is uncountable. Suppose to the contrary that Ω is countable. Applying Corollary 2.1, we then conclude that

$$1 = \sum_{\omega \in \Omega} P(\{\omega\}) = \sum_{\omega \in \Omega} 0 = 0,$$

a contradiction. Hence, Ω is uncountable.

2.3 Specifying Probabilities

Basic Exercises

2.37

a) An equal-likelihood model would be appropriate if and only if the six genotypes (*aa,ab,ao,bb,bo,oo*) are equally likely.

b) Let B denote the event that the person chosen has type B blood. Referring to Table 2.2, we see that $B = \{bb, bo\}$. Hence, assuming an equal-likelihood model, we have

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{2}{6} = \frac{1}{3}.$$

c) No, an equal-likelihood model is not appropriate because, as is well known, genotypes are not equally likely.

d) As we noted in part (b), we have $B = \{bb, bo\}$. Applying Proposition 2.2 on page 42 and referring to the table provided, we deduce that

$$P(B) = P(\{bb\}) + P(\{bo\}) = 0.007 + 0.116 = 0.123,$$

which is significantly smaller than the (incorrect) probability obtained in part (b).

2.41 Let us denote the four senators by r1, r2, d1, and d2, where r and d stand for Republican and Democrat, respectively. A sample space for the random experiment of selecting two of the four senators to constitute a subcommittee is $\Omega = \big\{\{\text{r1}, \text{r2}\}, \{\text{r1}, \text{d1}\}, \{\text{r1}, \text{d2}\}, \{\text{r2}, \text{d1}\}, \{\text{r2}, \text{d2}\}, \{\text{d1}, \text{d2}\}\big\}$. Because the selection is done at random, each of the six possible outcomes is equally likely. Thus, a classical probability model is appropriate here. So, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

Now, for $i = 0$, 1, and 2, let A_i denote the event that the number of Republicans on the subcommittee is exactly i.

a) We have $A_2 = \big\{\{\text{r1}, \text{r2}\}\big\}$ and, hence,

$$P(A_2) = \frac{N(A_2)}{6} = \frac{1}{6} \approx 0.167.$$

b) We have $A_1 = \big\{\{\text{r1}, \text{d1}\}, \{\text{r1}, \text{d2}\}, \{\text{r2}, \text{d1}\}, \{\text{r2}, \text{d2}\}\big\}$ and, hence,

$$P(A_1) = \frac{N(A_1)}{6} = \frac{4}{6} = \frac{2}{3} \approx 0.667.$$

c) We have $A_0 = \big\{\{\text{d1}, \text{d2}\}\big\}$ and, hence,

$$P(A_0) = \frac{N(A_0)}{6} = \frac{1}{6} \approx 0.167.$$

2.45 From Exercise 2.44, we know that $P(E) = |E|$ for each event E, where $|E|$ denotes the area of the set E. Also, see the note at the beginning of the solution to that exercise.

a) Let A denote the event that the x coordinate of the point chosen is less than the y coordinate. We note that A is the triangle with vertices (0, 0), (1, 0), and (1, 1). Hence, $P(A) = |A| = (1 \cdot 1)/2 = 1/2$.

b) Let B denote the event that the smaller of the two coordinates of the point chosen is less than 1/2. We note that event B^c is the rectangle with vertices $(1/2, 1/2)$, $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, we have

$$P(B) = |B| = |\Omega| - |B^c| = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

c) Let C denote the event that the smaller of the two coordinates of the point chosen exceeds 1/2. We note that C is the rectangle with vertices $(1/2, 1/2)$, $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, we have $P(C) = |C| = (1/2) \cdot (1/2) = 1/4$.

d) Let D denote the event that the larger of the two coordinates of the point chosen is less than 1/2. We note that D is the rectangle with vertices $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, and $(1/2, 0)$. Hence, we have $P(D) = |D| = (1/2) \cdot (1/2) = 1/4$.

e) Let E denote the event that the larger of the two coordinates of the point chosen exceeds 1/2. We note that event E^c is the rectangle with vertices $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, and $(1/2, 0)$. Hence, we have

$$P(E) = |E| = |\Omega| - |E^c| = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

f) Let F denote the event that the sum of the two coordinates is between 1 and 1.5. We note that event F^c consists of two triangles, one with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ and the other with vertices $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, we have

$$P(F) = |F| = |\Omega| - |F^c| = 1 - \left(\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{8}.$$

g) Let G denote the event that the sum of the two coordinates is between 1.5 and 2. We note that G is the triangle vertices $(1/2, 1)$, $(1, 1)$, and $(1, 1/2)$. Hence, $P(G) = |G| = (1/2) \cdot (1/2) \cdot (1/2) = 1/8$.

2.49 For definiteness, we position the side of the triangle from which the point is selected on the horizontal axis and center it at the origin. Then a sample space for the random experiment is $\Omega = [-\ell/2, \ell/2]$ and, as the point is selected at random, a geometric probability model is appropriate. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\big|[-\ell/2, \ell/2]\big|} = \frac{|E|}{\ell},$$

where $|E|$ denotes the length of the set E.

a) Let A_x denote the event that the distance of the point selected from the opposite vertex is at most x. We note that the height of the triangle is $\sqrt{3}\ell/2$. If $x < \sqrt{3}\ell/2$, then $A_x = \emptyset$ and, hence, $P(A_x) = 0$. If $x \geq \ell$, then $A_x = \Omega$ and, hence, $P(A_x) = 1$. Now assume that $\sqrt{3}\ell/2 \leq x < \ell$. From the Pythagorean theorem, we find that A_x is the interval with endpoints $\pm\sqrt{4x^2 - 3\ell^2}/2$. Hence,

$$P(A_x) = \frac{|E|}{\ell} = \frac{\sqrt{4x^2 - 3\ell^2}}{\ell}.$$

Consequently,

$$P(A_x) = \begin{cases} 0, & \text{if } x < \sqrt{3}\ell/2; \\ \sqrt{4x^2 - 3\ell^2}\big/\ell, & \text{if } \sqrt{3}\ell/2 \leq x < \ell; \\ 1, & \text{if } x \geq \ell. \end{cases}$$

b) Let B_{xy} denote the event that the distance of the point selected from the opposite vertex is between x and y. We assume, of course, that $x < y$. Then $A_x \subset A_y$ and, hence, from the additivity axiom,

$$P(A_y) = P\big((A_y \cap A_x) \cup (A_y \cap A_x^c)\big) = P(A_y \cap A_x) + P(A_y \cap A_x^c) = P(A_x) + P(A_y \cap A_x^c).$$

Noting that $B_{xy} = A_y \cap A_x^c$, we conclude, in view of part (a), that

$$P(B_{xy}) = P(A_y \cap A_x^c) = P(A_y) - P(A_x)$$
$$= \begin{cases} 0, & \text{if } y < \sqrt{3}\ell/2 \text{ or } x \ge \ell; \\ \sqrt{4y^2 - 3\ell^2}\big/\ell, & \text{if } x < \sqrt{3}\ell/2 \text{ and } \sqrt{3}\ell/2 \le y < \ell; \\ \left(\sqrt{4y^2 - 3\ell^2} - \sqrt{4x^2 - 3\ell^2}\right)\big/\ell, & \text{if } \sqrt{3}\ell/2 \le x < y < \ell; \\ 1 - \sqrt{4x^2 - 3\ell^2}\big/\ell, & \text{if } \sqrt{3}\ell/2 \le x < \ell \text{ and } y \ge \ell; \\ 1, & \text{if } x < \sqrt{3}\ell/2 \text{ and } y \ge \ell. \end{cases}$$

Advanced Exercises

2.53 A sample space for the random experiment is $\Omega = (0, 1)$. Because a number is being selected at random, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{|(0, 1)|} = \frac{|E|}{1} = |E|,$$

where $|E|$ denotes the length of the set E. Let A denote the event that the number selected is rational. Then $A = \mathcal{Q} \cap (0, 1)$, which is a countable set, being a subset of the countable set $\mathcal{Q}$. Let $r_1, r_2, \ldots$ be an enumeration of A. Then, from the additivity axiom,

$$P(A) = P\left(\bigcup_{n=1}^{\infty}\{r_n\}\right) = \sum_{n=1}^{\infty} P(\{r_n\}) = \sum_{n=1}^{\infty} |\{r_n\}| = \sum_{n=1}^{\infty}(r_n - r_n) = \sum_{n=1}^{\infty} 0 = 0.$$

2.57 Let x denote the distance from the center of the needle to the closest line and let y denote the angle, in radians, which the needle forms with that line. We note that the quantities x and y completely determine the position of the needle. Hence, a sample space for the experiment of dropping the needle onto the floor is $\Omega = \{\,(x, y) : 0 \le x \le d/2,\ 0 \le y \le \pi\,\}$. As the needle is dropped randomly, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\pi d/2} = \frac{2}{\pi d}|E|,$$

where $|E|$ denotes the area of the set E. Let A denote the event that the needle crosses one of the lines. We see that event A occurs if and only if $x \le (\ell/2)\sin y$. Hence, $A = \{\,(x, y) \in \Omega : x \le (\ell/2)\sin y\,\}$. Using calculus, we get

$$|A| = \iint_A dx\,dy = \int_0^{\pi}\left(\int_0^{(\ell/2)\sin y} dx\right)dy = \frac{\ell}{2}\int_0^{\pi}\sin y\,dy = \frac{\ell}{2}\cdot 2 = \ell.$$

Consequently,

$$P(A) = \frac{2}{\pi d}|A| = \frac{2}{\pi d}\cdot \ell = \frac{2\ell}{\pi d}.$$

2.4 Basic Properties of Probability

Basic Exercises

2.61 From Exercise 2.41, we know that choosing two senators at random from the four senators on the committee to form a subcommittee has six equally-likely outcomes. Let E denote the event that at least one Republican is on the subcommittee. We note that E^c is the event that no Republicans are on the subcommittee, that is, $E^c = \{\{\text{d1}, \text{d2}\}\}$. Hence, from the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - P\big(\{\{\text{d1}, \text{d2}\}\}\big) = 1 - \frac{1}{6} = \frac{5}{6} \approx 0.833.$$

By using the complementation rule, we need only obtain the probability of a simple event, whereas, if that rule were not used, we would have to obtain the probability of a compound event, that is, an event consisting of more than one outcome.

2.65 For a randomly selected U.S. adult, let F and D denote the events that the person obtained is a female and is divorced, respectively. We know that $P(F) = 0.510$, $P(D) = 0.071$, and $P(F \cap D) = 0.041$.
a) From the general addition rule,

$$P(F \cup D) = P(F) + P(D) - P(F \cap D) = 0.510 + 0.071 - 0.041 = 0.540.$$

b) Let M denote the event that the person obtained is a male. Applying the complementation rule, we get

$$P(M) = 1 - P(M^c) = 1 - P(F) = 1 - 0.510 = 0.490.$$

c) We want to determine $P(F \cap D^c)$. Applying the law of partitions, we get

$$0.510 = P(F) = P(D \cap F) + P(D^c \cap F) = 0.041 + P(D^c \cap F) = 0.041 + P(F \cap D^c).$$

Hence, $P(F \cap D^c) = 0.510 - 0.041 = 0.469$.
d) We want to determine $P(M \cap D)$. Applying the law of partitions, we get

$$0.071 = P(D) = P(F \cap D) + P(F^c \cap D) = 0.041 + P(M \cap D).$$

Hence, $P(M \cap D) = 0.071 - 0.041 = 0.030$.

2.69 We use the information and notation from Example 2.31.
a) Let A denote the event that the household selected gets either the *Times* or the *Herald*, but not both. We have $A = (T \cap H^c) \cup (T^c \cap H)$, and the two events in the union are mutually exclusive. From the law of partitions,

$$P\big(T \cap H^c\big) = P(T) - P(T \cap H) = 0.470 - 0.119 = 0.351$$

and

$$P\big(T^c \cap H\big) = P(H) - P(T \cap H) = 0.334 - 0.119 = 0.215.$$

Hence, from the additivity axiom,

$$P(A) = P\big((T \cap H^c) \cup (T^c \cap H)\big) = P\big(T \cap H^c\big) + P\big(T^c \cap H\big) = 0.351 + 0.215 = 0.566.$$

b) Let B denote the event that the household selected gets exactly one of the three newspapers. We have $B = (T \cap H^c \cap E^c) \cup (T^c \cap H \cap E^c) \cup (T^c \cap H^c \cap E)$, and the three events in the union are mutually exclusive. From the law of partitions,

$$P\big(T \cap H^c \cap E^c\big) = P\big(T \cap H^c\big) - P\big(T \cap H^c \cap E\big),$$
$$P\big(T \cap H^c\big) = P(T) - P(T \cap H),$$
$$P\big(T \cap H^c \cap E\big) = P(T \cap E) - P(T \cap H \cap E).$$

Hence,

$$\begin{aligned}P\big(T \cap H^c \cap E^c\big) &= \big(P(T) - P(T \cap H)\big) - \big(P(T \cap E) - P(T \cap H \cap E)\big)\\ &= P(T) - P(T \cap H) - P(T \cap E) + P(T \cap H \cap E)\\ &= 0.470 - 0.119 - 0.151 + 0.048\\ &= 0.248.\end{aligned}$$

Proceeding similarly, we find that $P(T^c \cap H \cap E^c) = 0.159$ and $P(T^c \cap H^c \cap E) = 0.139$. Therefore, by the additivity axiom,

$$\begin{aligned}P(B) &= P\big((T \cap H^c \cap E^c) \cup (T^c \cap H \cap E^c) \cup (T^c \cap H^c \cap E)\big)\\ &= P\big(T \cap H^c \cap E^c\big) + P\big(T^c \cap H \cap E^c\big) + P\big(T^c \cap H^c \cap E\big)\\ &= 0.248 + 0.159 + 0.139\\ &= 0.546.\end{aligned}$$

c) Let C denote the event that the household selected gets none of the three newspapers. Then we have $C = T^c \cap H^c \cap E^c$. Applying De Morgan's law, the complementation rule, and the result of Example 2.31, we get

$$P(C) = P\big(T^c \cap H^c \cap E^c\big) = P\big((T \cup H \cup E)^c\big) = 1 - P(T \cup H \cup E) = 1 - 0.824 = 0.176.$$

d) Let D denote the event that the household selected gets the *Times* and *Herald*, but not the *Examiner*. Then $D = T \cap H \cap E^c$. Hence, from the law of partitions,

$$P(D) = P\big(T \cap H \cap E^c\big) = P(T \cap H) - P(T \cap H \cap E) = 0.119 - 0.048 = 0.071.$$

e) For $k = 0, 1, 2, 3$, let E_k denote the event that the household selected gets exactly k of the three newspapers. We note that the E_ks form a partition of the sample space. Using the certainty axiom, the additivity axiom, and the results of parts (b) and (c), we deduce that

$$\begin{aligned}1 = P(\Omega) &= P(E_0 \cup E_1 \cup E_2 \cup E_3) = P(E_0) + P(E_1) + P(E_2) + P(E_3)\\ &= 0.176 + 0.546 + P(E_2) + 0.048.\end{aligned}$$

Hence, $P(E_2) = 1 - (0.176 + 0.546 + 0.048) = 0.230$.

Theory Exercises

2.73

a) As $B_1 = A_1$, we see that B_1 is the event that A_1 occurs. For $n \geq 2$, we have $B_n = A_n \cap A_{n-1}^c$, so that, in this case, B_n is the event that A_n occurs but A_{n-1} doesn't.

b) Let $m, n \in \mathcal{N}$ with $m \neq n$, say, $m < n$. Then, as $m \leq n - 1$, we have $A_m \subset A_{n-1}$. Therefore,

$$B_m \cap B_n = \big(A_m \cap A_{m-1}^c\big) \cap \big(A_n \cap A_{n-1}^c\big) \subset A_m \cap A_{n-1}^c \subset A_{n-1} \cap A_{n-1}^c = \emptyset.$$

Thus, $B_m \cap B_n = \emptyset$, and we have established that $B_1, B_2, \ldots$ are mutually exclusive.

Now, because $B_n \subset A_n$ for all $n \in \mathcal{N}$, we clearly have $\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$. Conversely, suppose that $\omega \in \bigcup_{n=1}^{\infty} A_n$. Then there is an $n \in \mathcal{N}$ such that $\omega \in A_n$. Let m be the smallest such n. Then $\omega \in A_m \cap A_{m-1}^c = B_m$ and, so, $\omega \in \bigcup_{n=1}^{\infty} B_n$. Therefore, $\bigcup_{n=1}^{\infty} B_n \supset \bigcup_{n=1}^{\infty} A_n$. Hence, we have shown that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

c) From the law of partitions and the fact that $A_{n-1} \subset A_n$, we deduce that, for $n \geq 2$,

$$P(B_n) = P(A_n \cap A_{n-1}^c) = P(A_n) - P(A_n \cap A_{n-1}) = P(A_n) - P(A_{n-1}).$$

d) From part (b), the additivity axiom, and part (c),

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n) = P(A_1) + \sum_{n=2}^{\infty}\big(P(A_n) - P(A_{n-1})\big)$$
$$= P(A_1) + \lim_{n\to\infty}\sum_{k=2}^{n}\big(P(A_k) - P(A_{k-1})\big) = P(A_1) + \lim_{n\to\infty}\big(P(A_n) - P(A_1)\big)$$
$$= \lim_{n\to\infty} P(A_n).$$

e) From parts (b) and (d),

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(A_n).$$

Hence, Proposition 2.11(a) holds.
f) Let $A_1, A_2, \dots$ be events such that $A_1 \supset A_2 \supset \cdots$. Setting $C_n = A_n^c$ for all $n \in \mathcal{N}$, we have that $C_1 \subset C_2 \subset \cdots$. Consequently, from De Morgan's law, the complementation rule, and Proposition 2.11(a), we get

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right) = 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = 1 - P\left(\bigcup_{n=1}^{\infty} C_n\right) = 1 - \lim_{n\to\infty} P(C_n)$$
$$= 1 - \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty}\big(1 - P(A_n^c)\big) = \lim_{n\to\infty} P(A_n).$$

Hence, Proposition 2.11(b) holds.

Advanced Exercises

2.77 From Exercise 2.18, we know that $A^* = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$. For each $n \in \mathcal{N}$, set $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $B_1 \supset B_2 \supset \cdots$. Hence, from the nonnegativity axiom, Proposition 2.11(b), Boole's inequality, and the fact that the tail of a convergent series converges to 0, we get that

$$0 \le P(A^*) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \le \lim_{n\to\infty}\sum_{k=n}^{\infty} P(A_k) = 0.$$

Therefore, $P(A^*) = 0$.

Review Exercises for Chapter 2

Basic Exercises

2.79 In the ordered-pair representation, the black die comes up 2 if and only if the first member of the ordered pair is 2. Hence, the event that the black die comes up 2 is

$$\{\,(2, y) : y \in \{1, 2, 3, 4, 5, 6\}\,\} = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\}.$$

2.80
a) The sample space for this random experiment consists of all subsets of size 2 of the six club members, where the elements of such a subset represent the two members selected as co-chairs. Hence,

$$\Omega = \big\{\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{a,f\},\{b,c\},\{b,d\},\{b,e\},\{b,f\},\{c,d\},\{c,e\},\{c,f\},\{d,e\},\{d,f\},\{e,f\}\big\}.$$

b) Let A denote the event that at least one of a and f is chosen as a co-chair. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$A = \big\{\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{a,f\},\{b,f\},\{c,f\},\{d,f\},\{e,f\}\big\}.$$

c) The sample space for this random experiment consists of all subsets of size 2 of the six club members in which one is a man and the other is a woman. The elements of such a subset represent the two members selected as co-chairs. Hence, in this case,

$$\Omega = \big\{\{a,d\},\{a,e\},\{a,f\},\{b,d\},\{b,e\},\{b,f\},\{c,d\},\{c,e\},\{c,f\}\big\}.$$

Let A denote the event that at least one of a and f is chosen as a co-chair. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$A = \big\{\{a,d\},\{a,e\},\{a,f\},\{b,f\},\{c,f\}\big\}.$$

d) The sample space for this random experiment consists of all ordered pairs from the six club members, where the first and second entries of such an ordered pair represent the chair and vice-chair selected, respectively. Hence,

$$\Omega = \big\{(a,b),(a,c),(a,d),(a,e),(a,f),(b,a),(b,c),(b,d),(b,e),(b,f),(c,a),(c,b),(c,d),(c,e),(c,f),(d,a),(d,b),(d,c),(d,e),(d,f),(e,a),(e,b),(e,c),(e,d),(e,f),(f,a),(f,b),(f,c),(f,d),(f,e)\big\}.$$

Let A denote the event that at least one of a and f is chosen as one of the two officers. As a subset of the sample space, A consists of all elements of Ω that contain either a or f (or both). Hence,

$$A = \big\{(a,b),(a,c),(a,d),(a,e),(a,f),(b,a),(b,f),(c,a),(c,f),(d,a),(d,f),(e,a),(e,f),(f,a),(f,b),(f,c),(f,d),(f,e)\big\}.$$

2.81 There are six possible categories in the cross classification, which we list in the following table:

Category, k	Description
1	foreign-made compact
2	foreign-made midsize
3	foreign-made fullsize
4	U.S.-made compact
5	U.S.-made midsize
6	U.S.-made fullsize

Now, let x_k denote the number of the 100 cars you observe that are of category k. Each possible outcome of the random experiment can then be represented as an ordered six-tuple, $(x_1, x_2, x_3, x_4, x_5, x_6)$. Hence, a sample space for this random experiment is

$$\Omega = \left\{ (x_1,x_2,x_3,x_4,x_5,x_6) : x_k \in \{0,1,2,\ldots\} \text{ for } 1 \le k \le 4, \text{ and } \textstyle\sum_{k=1}^{6} x_k = 100 \right\}.$$

2.82

a) The events that at least five men are chosen and at least five women are chosen are not mutually exclusive. Indeed, both events occur if five, six, or seven men are chosen.
b) The events that at least five men are chosen and at least eight women are chosen are mutually exclusive. Indeed, if at least five men are chosen, then at most seven women can be chosen.
c) The events that five men and seven women are chosen, four men and eight women are chosen, and three men and nine women are chosen are mutually exclusive. Indeed each of these three events specifies a different number of men (and women) that are chosen and, hence, no two of the events can occur simultaneously.
d) The events that the first person on the list of 20 is among those chosen, the second person on the list of 20 is among those chosen, and the third person on the list of 20 is among those chosen are not mutually exclusive. In fact, all three events can occur simultaneously, namely, if the first three people on the list are among the 12 people chosen for the jury.

2.83

a) A typical outcome of the random experiment of assigning the three alternates can be represented as an ordered triple, (x, y, z), where x, y, and z denote the first, second, and third alternates, respectively. Hence, a sample space is

$$\begin{aligned}\Omega &= \left\{ (x, y, z) : x, y, z \in \{\text{A, B, C}\} \text{ and } x \neq y \neq z \right\} \\ &= \{(\text{A, B, C}), (\text{A, C, B}), (\text{B, A, C}), (\text{B, C, A}), (\text{C, A, B}), (\text{C, B, A})\}.\end{aligned}$$

b) Person A is the first alternate if and only if the first entry of the ordered triple is A. Hence, as a subset of the sample space, the event that A is the first alternate is {(A, B, C), (A, C, B)}.
c) Event $A^c \cap B^c \cap C^c$ is that A is not the first alternate, B is not the second alternate, and C is not the third alternate. As a subset of the sample space, we have $A^c \cap B^c \cap C^c = \{(\text{B, C, A}), (\text{C, A, B})\}$.
d) Event $A \cap B \cap C$ is that A is the first alternate, B is the second alternate, and C is the third alternate. As a subset of the sample space, we have $A \cap B \cap C = \{(\text{A, B, C})\}$.
e) Event $A \cap C$ is that A is the first alternate and C is the third alternate. As a subset of the sample space, we have $A \cap C = \{(\text{A, B, C})\}$. We note that $A \cap B \cap C = A \cap C$.
f) Because who among A, B, and C will be the first alternate, who will be the second, and who will be the third will be decided by chance, an equal-likelihood model is appropriate here. Hence, each of the six possible outcomes should be assigned the same probability, namely, probability 1/6.
g) As an equal-likelihood model is appropriate, we have, for each event E, that

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

Referring now to parts (c)–(e), we get

$$\begin{aligned}P(A^c \cap B^c \cap C^c) &= \frac{N(A^c \cap B^c \cap C^c)}{6} = \frac{N\big(\{(\text{B, C, A}), (\text{C, A, B})\}\big)}{6} = \frac{2}{6} = \frac{1}{3}, \\ P(A \cap B \cap C) &= \frac{N(A \cap B \cap C)}{6} = \frac{N\big(\{(\text{A, B, C})\}\big)}{6} = \frac{1}{6}, \\ P(A \cap C) &= \frac{N(A \cap C)}{6} = \frac{N\big(\{(\text{A, B, C})\}\big)}{6} = \frac{1}{6}.\end{aligned}$$

2.84 Answers will vary. Following is what we did and got. In each of parts (a)–(c), we used a random-number generator to obtain 1000 random decimal digits; in other words, in each part, we simulated 1000 times the random experiment of selecting a number at random from the set $\{0, 1, 2, \ldots, 9\}$.
a) In this simulation, event A (0, 1, 2, or 3) occurred 395 times and, hence, the proportion of times that event A occurred is 0.395. In view of the frequentist interpretation of probability, this proportion estimates $P(A)$.
b) In this simulation, event B (7 or 8) occurred 191 times and, hence, the proportion of times that event B occurred is 0.191. In view of the frequentist interpretation of probability, this proportion estimates $P(B)$.
c) In this simulation, event $A \cup B$ (0, 1, 2, 3, 7, or 8) occurred 608 times and, hence, the proportion of times that event $A \cup B$ occurred is 0.608. In view of the frequentist interpretation of probability, this proportion estimates $P(A \cup B)$.
d) From the additivity axiom, $P(A \cup B) = P(A) + P(B)$ and, from our estimates in parts (a)–(c), $P(A \cup B) \approx 0.608$ and $P(A) + P(B) \approx 0.395 + 0.191 = 0.586$. Hence, for these simulations, the results are consistent with the additivity axiom provided that we round to one decimal place. Given that the number of simulations is only moderate, the discrepancy here is not surprising.
e) Repeating parts (a)–(c), but with 10,000 simulations each time, we obtained $P(A \cup B) \approx 0.5961$ and $P(A) + P(B) \approx 0.3977 + 0.2009 = 0.5986$. Hence, for these simulations, the results are consistent with the additivity axiom provided that we round to two decimal places.

2.85 The completed table is as follows:

Event	Description
$A \cap B^c$	A occurs but B doesn't.
$A \cup B^c$	Either A occurs or B doesn't.
$A^c \cap B^c$	Neither A nor B occurs.
$(A \cap B^c) \cup (A^c \cap B)$	Exactly one of A and B occurs.
$A \cap B \cap C$	Events A, B, and C occur.
$A \cap (B \cup C)$	A occurs and either B or C occurs.
$A \cup (B \cap C)$	Either A occurs or both B and C occur.
$A \cup B \cup C$	At least one of A, B, and C occurs.
$A^c \cup B^c \cup C^c$	At least one of A, B, and C does not occur.
$\bigcap_n A_n$	All of $A_1, A_2, \ldots$ occur.
$\bigcup_n A_n$	At least one of $A_1, A_2, \ldots$ occurs.

2.86 Exactly one of event A and event B occurs if and only if either event A occurs and event B doesn't or event B occurs and event A doesn't, which is the case if and only if either both event A and event B^c occur or both event B and event A^c occur, that is, if and only if event $(A \cap B^c) \cup (B \cap A^c)$ occurs. Other ways to express the event in question are possible. For instance, that event occurs if and only if at least one but not both of event A and event B occur, which is the event $(A \cup B) \cap (A^c \cup B^c)$.

2.87
a) True. If A, B, and C are (pairwise) mutually exclusive, then, in particular, we have $A \cap B = \emptyset$, which means that A and B are mutually exclusive.
b) False. It's possible for A and B to be mutually exclusive without A, B, and C being (pairwise) mutually exclusive. For instance, any three events, A, B, and C, such that $A \cap B = \emptyset$ and $A \cap C \neq \emptyset$ will provide such a situation.

2.88

a) Because $0 \le p \le 1$, the numbers p^2, $p(1-p)$, $p(1-p)$, and $(1-p)^2$ are nonnegative. They also sum to 1 because

$$\begin{aligned} 1 = 1^2 &= \big(p + (1-p)\big)^2 = p^2 + 2p(1-p) + (1-p)^2 \\ &= p^2 + p(1-p) + p(1-p) + (1-p)^2. \end{aligned}$$

Therefore, from Proposition 2.3 on page 43, the specified assignment determines a unique probability measure on Ω.

b) Referring to Proposition 2.2 on page 42, we see, for instance, that

$$\begin{aligned} P(\{\text{HH}, \text{HT}, \text{TH}\}) &= P(\{\text{HH}\}) + P(\{\text{HT}\}) + P(\{\text{TH}\}) = p^2 + p(1-p) + p(1-p) \\ &= p\big(p + (1-p) + (1-p)\big) = p(2-p). \end{aligned}$$

Proceeding similarly, we obtain the following table:

Event	Probability	Event	Probability
$\emptyset$	0	{HT, TH}	$2p(1-p)$
{HH}	p^2	{HT, TT}	$1-p$
{HT}	$p(1-p)$	{TH, TT}	$1-p$
{TH}	$p(1-p)$	{HH, HT, TH}	$p(2-p)$
{TT}	$(1-p)^2$	{HH, HT, TT}	$1-p+p^2$
{HH, HT}	p	{HH, TH, TT}	$1-p+p^2$
{HH, TH}	p	{HT, TH, TT}	$1-p^2$
{HH, TT}	$1-2p+2p^2$	Ω	1

2.89

a) We can represent a typical outcome of the random experiment as a triple (x_1, x_2, x_3), where each x_k is either w or b depending on whether component k is working or broken, respectively. Hence, a sample space is

$$\begin{aligned} \Omega &= \big\{\, (x_1, x_2, x_3) : x_k \in \{w, b\} \text{ for } k = 1, 2, 3 \,\big\} \\ &= \{(w, w, w), (w, w, b), (w, b, w), (w, b, b), (b, w, w), (b, w, b), (b, b, w), (b, b, b)\}. \end{aligned}$$

As each component is equally likely to be working or broken, each of the eight possible outcomes is equally likely. Thus, a classical probability model is appropriate here and we have, for each event E, that

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{8}.$$

b) Let A denote the event that all three components are broken. Then $A = \{(b, b, b)\}$ and

$$P(A) = P\big(\{(b, b, b)\}\big) = \frac{N\big(\{(b, b, b)\}\big)}{8} = \frac{1}{8}.$$

c) Let B denote the event that the device is functioning, that is, that at least two of the three components are working. Then $B = \{(w, w, w), (w, w, b), (w, b, w), (b, w, w)\}$ and

$$\begin{aligned} P(B) &= P\big(\{(w, w, w), (w, w, b), (w, b, w), (b, w, w)\}\big) \\ &= \frac{N\big(\{(w, w, w), (w, w, b), (w, b, w), (b, w, w)\}\big)}{8} = \frac{4}{8} = \frac{1}{2}. \end{aligned}$$

2.90 The sample space, Ω, is the unit disk and, because a point is being chosen at random, a geometric probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{\pi},$$

where $|E|$ denotes the area of the set E.
a) Let A denote the event that the distance from the point chosen to the center of the disk exceeds 1/2. Then event A^c is that the distance from the point chosen to the center of the disk is at most 1/2. Thus, A^c is the disk of radius $1/2$ centered at the origin, which has area $\pi \cdot (1/2)^2 = \pi/4$. Consequently, from the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{\pi} = 1 - \frac{\pi/4}{\pi} = \frac{3}{4}.$$

b) Let B denote the event that the sum of the magnitudes of the two coordinates of the point chosen exceeds 1/2. Then event B^c is that the sum of the magnitudes of the two coordinates of the point chosen is at most 1/2. Thus, B^c is the square with vertices (1/2, 0), (0, 1/2), $(-1/2, 0)$, and $(0, -1/2)$, which has area $\left(1/\sqrt{2}\right)^2 = 1/2$. Consequently, from the complementation rule,

$$P(B) = 1 - P(B^c) = 1 - \frac{|B^c|}{\pi} = 1 - \frac{1/2}{\pi} = 1 - \frac{1}{2\pi} \approx 0.841.$$

c) Let C denote the event that the maximum of the magnitudes of the two coordinates of the point chosen exceeds 1/2. Then event C^c is that the maximum of the magnitudes of the two coordinates of the point chosen is at most 1/2. Thus, C^c is the square with vertices $(1/2, 1/2)$, $(-1/2, 1/2)$, $(-1/2, -1/2)$, and $(1/2, -1/2)$, which has area $1^2 = 1$. Consequently, from the complementation rule,

$$P(C) = 1 - P(C^c) = 1 - \frac{|C^c|}{\pi} = 1 - \frac{1}{\pi} \approx 0.682.$$

2.91
a) We have $1/2^n \geq 0$ for all $n \in \mathcal{N}$ and, moreover, from the formula for a geometric series,

$$\sum_{n \in \mathcal{N}} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1.$$

Hence, from Proposition 2.3 on page 43, the specified probability assignment is legitimate.
b) Let O denote the event that the outcome is an odd number. Then $O = \{1, 3, 5, \ldots\}$ and, hence, from Proposition 2.2 on page 42 and the formula for a geometric series,

$$P(O) = \sum_{\omega \in O} P(\{\omega\}) = \sum_{n=0}^{\infty} P(\{2n+1\}) = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{2}{3}.$$

2.92
a) If event B_6 occurs, then, after the sixth toss, you have still not reached your goal, which implies that, after the fifth toss, you have still not reached your goal, that is, that event B_5 occurs. Hence, $B_6 \subset B_5$ and, therefore, from the domination principle, we have $P(B_6) \leq P(B_5)$.
b) Let A denote the event that you never reach your goal. We note that event A occurs if and only if after the nth toss, you have still not reached your goal for each $n \geq 3$, that is, if and only if event B_n occurs for all $n \geq 3$. Thus, $A = \bigcap_{n=3}^{\infty} B_n$. Arguing as in part (a), we see that $B_3 \supset B_4 \supset \cdots$. Therefore, from

the continuity property of a probability measure, specifically, Proposition 2.11(b) on page 74, we have

$$P(A) = P\left(\bigcap_{n=3}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n).$$

c) Let A denote the event that you never reach your goal. We note that, if either the first three tosses are heads or the first toss is a tail followed by five consecutive heads, then you reach your goal, that is, event A^c occurs. The probability that the first three tosses are heads is $(1/2)^3 = 1/8$ and the probability that the first toss is a tail followed by five consecutive heads is $(1/2)^6 = 1/64$. Hence, from the complementation rule and the domination principle,

$$P(A) = 1 - P(A^c) \leq 1 - \left(\frac{1}{8} + \frac{1}{64}\right) < 1 - \frac{1}{8} = \frac{7}{8}.$$

2.93 Let us measure time in hours after 3:00 P.M. A typical outcome of the random experiment can be represented as an ordered pair (x, y), where x and y represent the arrival times of woman 1 and woman 2, respectively. Thus, a sample space is $\Omega = [0, 2] \times [0, 2]$. Moreover, because each arrival time is equally likely between 3:00 P.M. and 5:00 P.M. and the time of arrival of one woman doesn't affect that of the other, a geometric probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{2 \cdot 2} = \frac{|E|}{4},$$

where $|E|$ is the area of the set E. Now, let A denote the event that the two women meet. Then we have $A = \{\,(x, y) \in \Omega : |x - y| \leq 2/3$. We see that A^c consists of (the interior of) two triangles, one with vertices $(0, 2/3)$, $(0, 2)$, and $(4/3, 2)$, and the other with vertices $(2/3, 0)$, $(2, 0)$, and $(2, 4/3)$. Hence, we have $|A^c| = 2 \cdot (4/3)^2/2 = 16/9$. Consequently, by the complementation rule,

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{4} = 1 - \frac{16/9}{4} = \frac{5}{9}.$$

2.94 Estimating the probability of rain (and, more generally, weather prediction) is a complex issue. Answers will vary among all three methods for specifying probabilities—probability models, empirical probability, and subjective probability—and combinations thereof.

2.95

a) Empirical probability. Specifically, we would observe a large number of bolts produced by the manufacturing process and use the proportion of those bolts that are defective as (our estimate of) the probability of a defective bolt.

b) Answers will vary. Some might say that subjective probability, based on an educated guess or intuition, is the only way to assign the probability of a specified horse finishing first in a particular horse race. Others might argue that empirical probability, based, say, on information from the racing form, can be used. Still others might claim that a probability model, employing the win odds (obtained from the totalizator board) can be used.

c) Empirical probability. Specifically, we would observe a large number of horse races and use the proportion of those races in which the favorite wins as (our estimate of) the probability that the favorite in a horse race will finish first.

d) Answers will vary. See the solution to Exercise 2.94.

2.96 For a randomly selected household in this town, let A and H denote the events that the household chosen owns an automobile and owns a home, respectively. We know that $P(A) = 0.80$, $P(H) = 0.45$, and $P(A \cap H) = 0.35$.
a) The event that the household chosen owns either an automobile or a home but not both can be expressed as the union of two mutually exclusive events as follows: $(A \cap H^c) \cup (H \cap A^c)$. From the law of partitions,

$$P(A \cap H^c) = P(A) - P(A \cap H) = 0.80 - 0.35 = 0.45$$

and

$$P(H \cap A^c) = P(H) - P(H \cap A) = 0.45 - 0.35 = 0.10.$$

Therefore, from the additivity axiom,

$$P\big((A \cap H^c) \cup (H \cap A^c)\big) = P(A \cap H^c) + P(H \cap A^c) = 0.45 + 0.10 = 0.55.$$

b) The event that the household chosen owns neither an automobile nor a home can be expressed as $A^c \cap H^c$. From the general addition rule,

$$P(A \cup H) = P(A) + P(H) - P(A \cap H) = 0.80 + 0.45 - 0.35 = 0.90.$$

Therefore, from De Morgan's law and the complementation rule,

$$P\big(A^c \cap H^c\big) = P\big((A \cup H)^c\big) = 1 - P(A \cup H) = 1 - 0.90 = 0.10.$$

2.97 For the table presented in the problem statement, we sum each row, each column, and all the cell entries to obtain the following table:

		Country			
		U.S. C_1	Canada C_2	Mexico C_3	Total
Vehicle type	Automobiles V_1	129,728	13,138	8,607	151,473
	Motorcycles V_2	3,871	320	270	4,461
	Trucks V_3	75,940	6,933	4,287	87,160
	Total	209,539	20,391	13,164	243,094

In what follows, all frequencies are in thousands.
a) The number of vehicles that are not automobiles is $4{,}461 + 87{,}160 = 91{,}621$, obtained by summing the total number of motorcycles and trucks.
b) From the second-column total, we see that the number of Canadian vehicles is 20,391.
c) From the second-row total, we see that the number of motorcyles is 4461.
d) The cell in the second row and second column shows that the number of Canadian motorcycles is 320.
e) Summing the second-column total and second-row total and then subtracting the number of Canadian motorcyles (which is counted twice in summing the two forementioned totals), we find that the number of vehicles that are either Canadian or motorcycles is $20{,}391 + 4{,}461 - 320 = 24{,}532$.
f) C_1 is the event that the vehicle selected is in the United States; V_3 is the event that the vehicle selected is a truck; and $C_1 \cap V_3$ is the event that the vehicle selected is a United States truck.

Note: Because a North American vehicle is being selected at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{243{,}094}.$$

g) We have

$$P(C_1) = \frac{N(C_1)}{243{,}094} = \frac{209{,}539}{243{,}094} \approx 0.862,$$

$$P(V_3) = \frac{N(V_3)}{243{,}094} = \frac{87{,}160}{243{,}094} \approx 0.359,$$

$$P(C_1 \cap V_3) = \frac{N(C_1 \cap V_3)}{243{,}094} = \frac{75{,}940}{243{,}094} \approx 0.312.$$

h) From the table presented at the beginning of the solution to this exercise, we find that the number of vehicles that are either in the United States or are trucks is $209{,}539 + 87{,}160 - 75{,}940 = 220{,}759$. Hence,

$$P(C_1 \cup V_3) = \frac{N(C_1 \cup V_3)}{243{,}094} = \frac{220{,}759}{243{,}094} \approx 0.908.$$

i) Applying the general addition rule to the results of part (g), we get

$$P(C_1 \cup V_3) = P(C_1) + P(V_3) - P(C_1 \cap V_3) \approx 0.862 + 0.359 - 0.312 = 0.909.$$

The discrepancy between the answer here and that in part (h) is due to roundoff error.

j) Dividing each entry of the table presented at the beginning of the solution to this exercise by the grand total of 243,094, we obtain the following joint probability distribution:

Country

Vehicle type	U.S. C_1	Canada C_2	Mexico C_3	$P(V_i)$
Automobiles V_1	0.534	0.054	0.035	0.623
Motorcycles V_2	0.016	0.001	0.001	0.018
Trucks V_3	0.312	0.029	0.018	0.359
$P(C_j)$	0.862	0.084	0.054	1.000

2.98 A typical outcome of this random experiment can be represented as an ordered triple, (x_1, x_2, x_3), where x_k denotes the number of the ball obtained on draw k. Hence, a sample space is

$$\Omega = \left\{ (x_1, x_2, x_3) : x_k \in \{1, 2, 3\} \text{ for } k = 1, 2, 3, \text{ and } x_1 \neq x_2 \neq x_3 \right\}$$
$$= \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}.$$

Because the balls are selected at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

a) We have

$$P(A_1) = P\big(\{(1, 2, 3), (1, 3, 2)\}\big) = \frac{N\big(\{(1, 2, 3), (1, 3, 2)\}\big)}{6} = \frac{2}{6} = \frac{1}{3},$$
$$P(A_2) = P\big(\{(1, 2, 3), (3, 2, 1)\}\big) = \frac{N\big(\{(1, 2, 3), (3, 2, 1)\}\big)}{6} = \frac{2}{6} = \frac{1}{3},$$
$$P(A_3) = P\big(\{(1, 2, 3), (2, 1, 3)\}\big) = \frac{N\big(\{(1, 2, 3), (2, 1, 3)\}\big)}{6} = \frac{2}{6} = \frac{1}{3}.$$

Note: A much simpler way to obtain these probabilities is to use a *symmetry argument.* Specifically, as the balls are selected at random, the ith ball chosen is equally likely to be any one of the three balls; hence, the probability is $1/3$ that it will be ball i. Thus, we have $P(A_i) = 1/3$ for $i = 1, 2$, and 3.
b) Events A_1, A_2, and A_3 are not mutually exclusive. In fact, the outcome $(1, 2, 3)$ is common to all three of those events.

2.99 A sample space for this random experiment is

$$\Omega = \{\text{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}\}.$$

Because the coin is balanced, each of the 16 possible outcomes are equally likely. Hence, a classical probability model is appropriate here and we have, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{16}.$$

a) The event that the first tail is followed by two consecutive heads is {HHHH, HTHH, THHH, THHT} and, hence, the probability of that event is $4/16 = 1/4$. Note that the outcome HHHH vacuously satisfies the condition that the first tail is followed by two consecutive heads.
b) The event that a run of three or more heads occurs is {HHHH, HHHT, THHH} and, hence, the probability of that event is $3/16$.

2.100 A typical outcome of this random experiment can be represented as an ordered triple, (x_1, x_2, x_3), where x_k denotes the number of the husband with which wife k dances. Hence, a sample space is

$$\Omega = \left\{ (x_1, x_2, x_3) : x_k \in \{1, 2, 3\} \text{ for } k = 1, 2, 3, \text{ and } x_1 \neq x_2 \neq x_3 \right\}$$
$$= \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}.$$

Because the couples are paired at random, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6}.$$

a) The event that each wife dances with her own husband is $\{(1, 2, 3)\}$ and, hence, the probability of that event is $1/6$.
b) The event that no wife dances with her own husband is $\{(2, 3, 1), (3, 1, 2)\}$ and, hence, the probability of that event is $2/6 = 1/3$.
c) The event that at least one wife dances with her own husband is the complement of the event that no wife dances with her own husband. Hence, from the complementation rule and part (b), the required probability is $1 - 1/3 = 2/3$.

2.101 Let W, D, and I denote the events that the customer will purchase a washer, dryer, and iron, respectively. We know that

$$P(W) = 0.4, \qquad P(D) = 0.3, \qquad P(I) = 0.23,$$
$$P(W \cap D) = 0.15, \qquad P(W \cap I) = 0.13, \qquad P(D \cap I) = 0.09,$$
$$P(W \cap D \cap I) = 0.05.$$

a) Let A denote the event that the customer purchases none of the three items. Then $A = W^c \cap D^c \cap I^c$. From the inclusion–exclusion principle,

$$\begin{aligned} P(W \cup D \cup I) &= P(W) + P(D) + P(I) - P(W \cap D) - P(W \cap I) - P(D \cap I) + P(W \cap D \cap I) \\ &= 0.4 + 0.3 + 0.23 - 0.15 - 0.13 - 0.09 + 0.05 = 0.61. \end{aligned}$$

Hence, from De Morgan's law and the complementation rule, we have

$$P(A) = P\big(W^c \cap D^c \cap I^c\big) = P\big((W \cup D \cup I)^c\big) = 1 - P(W \cup D \cup I) = 1 - 0.61 = 0.39.$$

b) Let B denote the event that the customer purchases two or more items. We note that event B occurs if and only if the customer purchases both a washer and a dryer or both a washer and an iron or both a dryer and an iron. Hence, $B = (W \cap D) \cup (W \cap I) \cup (D \cap I)$. Observing that $(W \cap D) \cap (W \cap I)$, $(W \cap D) \cap (D \cap I)$, $(W \cap I) \cap (D \cap I)$, and $(W \cap D) \cap (W \cap I) \cap (D \cap I)$ all equal $W \cap D \cap I$, we can apply the inclusion–exclusion principle to get

$$\begin{aligned} P(B) &= P\big((W \cap D) \cup (W \cap I) \cup (D \cap I)\big) \\ &= P(W \cap D) + P(W \cap I) + P(D \cap I) - 3P(W \cap D \cap I) + P(W \cap D \cap I) \\ &= 0.15 + 0.13 + 0.09 - 2 \cdot 0.05 = 0.27. \end{aligned}$$

c) Refer to parts (a) and (b). Let C denote the event that the customer purchases exactly one of the items. We note that event C^c occurs if and only if the customer purchases either no items (event A) or two or more items (event B); in other words, $C^c = A \cup B$. Hence, from the complementation rule, the additivity axiom, and the results of parts (a) and (b),

$$P(C) = 1 - P(C^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) = 1 - 0.39 - 0.27 = 0.34.$$

2.102 Let N and L denote the events that, during any given hour, you receive a nonlegitimate e-mail message and a legitimate e-mail message, respectively. From the problem statement, we have $P(N) = 0.5$, $P(L) = 0.7$, and $P(N \cap L) = 0.4$. The event that you receive no e-mail message during a given hour is $N^c \cap L^c$. Hence, from De Morgan's law, the complementation rule, and the general addition rule,

$$\begin{aligned} P\big(N^c \cap L^c\big) &= P\big((N \cup L)^c\big) = 1 - P(N \cup L) = 1 - \big(P(N) + P(L) - P(N \cap L)\big) \\ &= 1 - (0.5 + 0.7 - 0.4) = 0.2. \end{aligned}$$

Theory Exercises

2.103 We want to use mathematical induction to prove that, for all $N \in \mathcal{N}$,

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n).$$

Equation (2.4) implies that this result holds for $N = 2$. Assuming its truth for N, we prove it for $N + 1$. So, assume that $A_1, \ldots, A_{N+1}$ are mutually exclusive events. Then $A_1, \ldots, A_N$ are mutually exclusive.

Furthermore, from the distributive law for sets,

$$\left(\bigcup_{n=1}^{N} A_n\right) \cap A_{N+1} = \bigcup_{n=1}^{N} (A_n \cap A_{N+1}) = \bigcup_{n=1}^{N} \emptyset = \emptyset,$$

so that $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are mutually exclusive. Therefore, from Equation (2.4) and the induction assumption,

$$P\left(\bigcup_{n=1}^{N+1} A_n\right) = P\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) = P\left(\bigcup_{n=1}^{N} A_n\right) + P(A_{N+1})$$
$$= \sum_{n=1}^{N} P(A_n) + P(A_{N+1}) = \sum_{n=1}^{N+1} P(A_n),$$

as required. It is still necessary to assume Equation (2.5) instead of simply Equation (2.4) because we need additivity to hold for a countably infinite number of mutually exclusive events—and that condition does not follow from finite additivity.

2.104

a) If events A and B both have probability 0, then, by the nonnegativity axiom and the general addition rule,

$$0 \le P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B) = 0 + 0 = 0.$$

Hence, $P(A \cup B) = 0$.

b) If events A and B both have probability 1, then, from the complementation rule, A^c and B^c both have probability 0. Therefore, from the complementation rule, De Morgan's law, and part (a),

$$P(A \cap B) = 1 - P\big((A \cap B)^c\big) = 1 - P(A^c \cup B^c) = 1 - 0 = 1.$$

c) We prove that parts (a) and (b) hold for countably many events. For part (a), let $A_1, A_2, \ldots$ be events, all of which have probability 0. Then, by the nonnegativity axiom and Boole's inequality (Exercise 2.75),

$$0 \le P\left(\bigcup_n A_n\right) \le \sum_n P(A_n) = \sum_n 0 = 0.$$

Hence, $P\big(\bigcup_n A_n\big) = 0$. For part (b), let $A_1, A_2, \ldots$ be events, all of which have probability 1. Then, by the complementation rule, $A_1^c, A_2^c, \ldots$ all have probability 0. Therefore, from the complementation rule, De Morgan's law, and the result we just proved,

$$P\left(\bigcap_n A_n\right) = 1 - P\left(\left(\bigcap_n A_n\right)^c\right) = 1 - P\left(\bigcup_n A_n^c\right) = 1 - 0 = 1.$$

d) We will provide counterexamples in both cases. Let $\Omega = [0, 1]$, endowed with a geometric probability model, so that, for each event E,

$$P(E) = \frac{|E|}{|\Omega|} = \frac{|E|}{1} = |E|,$$

where E denotes the length of the set E. For each $\omega \in \Omega$, let $A_\omega = \{\omega\}$. Then, $P(A_\omega) = |\{\omega\}| = 0$ for all $\omega \in \Omega$. However,

$$P\left(\bigcup_{\omega\in\Omega} A_\omega\right) = P\left(\bigcup_{\omega\in\Omega} \{\omega\}\right) = P(\Omega) = 1 \ne 0.$$

Hence, we have found an uncountable collection of events with the property that each event has probability 0, but the union of the events does not have probability 0. Now let $B_\omega = A_\omega^c$ for each $\omega \in \Omega$. From the complementation rule, each B_ω has probability 1. However, the intersection of the B_ωs is empty and, hence, has probability 0, not 1.

Advanced Exercises

2.105 In each part, we will provide two solutions, one that uses formal techniques developed in this chapter and the other that applies a slicker approach. Let A denote the event that the second card drawn is an ace. Also, for convenience, we label the cards with the numbers 1–52, where the first four cards are the aces. Let $S = \{1, 2, \ldots, 52\}$.

a) A typical outcome for this random experiment can be represented as an ordered pair, (x, y), where x and y are the numbers of the first and second cards drawn, respectively. Hence, because the first card is replaced before the second card is drawn, a sample space is $\Omega = \{\,(x, y) : x, y \in S\,\}$. As the cards are randomly drawn, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{52 \cdot 52} = \frac{N(E)}{2704}.$$

Now,

$$A = \left\{\,(x, y) : x \in S,\ \ y \in \{1, 2, 3, 4\}\,\right\}.$$

Consequently,

$$P(A) = \frac{N(A)}{2704} = \frac{52 \cdot 4}{2704} = \frac{1}{13}.$$

A slicker solution is as follows: Because the first card is replaced in the deck before the second card is drawn, the second draw behaves exactly like the first draw. Hence, the probability that the second card drawn is an ace is the same as the probability that the first card drawn is an ace, which is $4/52 = 1/13$.

b) A typical outcome for this random experiment can be represented as an ordered pair, (x, y), where x and y are the numbers of the first and second cards drawn, respectively. Hence, because the first card is not replaced before the second card is drawn, a sample space is $\Omega = \{\,(x, y) : x, y \in S \text{ and } y \neq x\,\}$. As the cards are randomly drawn, a classical probability model is appropriate here. Thus, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{52 \cdot 52 - 52} = \frac{N(E)}{52 \cdot 51} = \frac{N(E)}{2652}.$$

Now,

$$A = \left\{\,(x, y) : x \in S,\ \ y \in \{1, 2, 3, 4\},\ \text{and } y \neq x\,\right\}.$$

Consequently,

$$P(A) = \frac{N(A)}{2652} = \frac{52 \cdot 4 - 4}{2652} = \frac{51 \cdot 4}{2652} = \frac{1}{13}.$$

A slicker solution is as follows: By symmetry, the second card drawn is equally likely to be any one of the 52 cards. Hence, the probability that the second card drawn is an ace is $4/52 = 1/13$.

2.106

a) From Exercise 1.62(b), we have $A \setminus B = A \cap B^c$, which is the event that A occurs but B doesn't.

b) From the law of partitions, we get that

$$P(A \setminus B) = P(A \cap B^c) = P(A) - P(A \cap B).$$

c) If $B \subset A$, then $A \cap B = B$. Hence, from part (b),

$$P(A \setminus B) = P(A) - P(A \cap B) = P(A) - P(B).$$

2.107

a) From the solution to Exercise 1.63, we know that $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Consequently, in view of the solution to Exercise 2.106(a), we see that $A \triangle B$ is the event that either A occurs but B doesn't or B occurs but A doesn't. In other words, $A \triangle B$ is the event that exactly one of A and B occurs.
b) We note that $A \setminus B$ and $B \setminus A$ are mutually exclusive events. Hence, from part (a), the additivity axiom, and Exercise 2.106(b), we get

$$\begin{aligned} P(A \triangle B) &= P\big((A \setminus B) \cup (B \setminus A)\big) = P(A \setminus B) + P(B \setminus A) \\ &= \big(P(A) - P(A \cap B)\big) + \big(P(B) - P(A \cap B)\big) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

2.108

a) We have that $\omega \in \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k\right)$ if and only if there is an $n \in \mathcal{N}$ such that $\omega \in \bigcap_{k=n}^{\infty} A_k$, that is, if and only if there is an $n \in \mathcal{N}$ such that $\omega \in A_k$ for all $k \geq n$, which is the case if and only if ω is in all but finitely many of the A_ns. Therefore, we see that $\bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k\right)$ is the event that all but finitely many of the A_ns occur.
b) We have that $\omega \in \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)$ if and only if $\omega \in \bigcup_{k=n}^{\infty} A_k$ for each $n \in \mathcal{N}$, that is, if and only if for each $n \in \mathcal{N}$ there is a $k \geq n$ such that $\omega \in A_k$, which is the case if and only if ω is in infinitely many of the A_ns. Therefore, we see that $\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)$ is the event that infinitely many of the A_ns occur.
c) If all but finitely many of the A_ns occur, then infinitely many of the A_ns occur. Hence, in view of parts (a) and (b), we have $\bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k\right) \subset \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)$.

2.109 From the domination principle and the certainty axiom, we have $P(E) \leq P(\Omega) = 1$ for all events E; thus, $P(E) \leq 1$ for all events E. For $1 \leq k \leq 8$, let A_k denote the event of a "5" on the kth throw of the die and set $A = \bigcup_{k=1}^{8} A_k$. Your colleague's reasoning shows that $P(A) = 1.333\ldots$, which is impossible because an event must have probability at most 1. Your colleague's reasoning is faulty because he or she used the additivity axiom on the non–mutually exclusive events $A_1, \ldots, A_8$.

2.110 A typical outcome of the random experiment of throwing a die three times can be represented as a triple (x_1, x_2, x_3), where x_k denotes the result of the kth throw. Hence, a sample space is

$$\Omega = \left\{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\}.$$

Because the die is balanced, a classical probability model is appropriate here. Hence, for each event E,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(E)}{6 \cdot 6 \cdot 6} = \frac{N(E)}{216}.$$

For $1 \leq n \leq 3$, let A_n denote the event of a "5" on the nth throw of the die and set $A = \bigcup_{n=1}^{3} A_n$. We want to determine $P(A)$. Now,

$$\begin{aligned} P(A_1) &= P\Big(\left\{ (5, x_2, x_3) : x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\}\Big) \\ &= \frac{N\Big(\left\{ (5, x_2, x_3) : x_2, x_3 \in \{1, 2, 3, 4, 5, 6\} \right\}\Big)}{216} \\ &= \frac{6 \cdot 6}{216} = \frac{1}{6}, \end{aligned}$$

and, likewise, $P(A_2) = P(A_3) = 1/6$. Also,

$$\begin{aligned} P(A_1 \cap A_2) &= P\Big(\big\{\,(5,5,x_3) : x_3 \in \{1,2,3,4,5,6\}\,\big\}\Big) \\ &= \frac{N\Big(\big\{\,(5,5,x_3) : x_3 \in \{1,2,3,4,5,6\}\,\big\}\Big)}{216} \\ &= \frac{6}{216} = \frac{1}{36}, \end{aligned}$$

and, likewise, $P(A_1 \cap A_3) = P(A_2 \cap A_3) = 1/36$. Also,

$$P(A_1 \cap A_2 \cap A_3) = P\big(\{(5,5,5)\}\big) = \frac{1}{216}.$$

Hence, from the inclusion–exclusion principle,

$$\begin{aligned} P(A) &= P\left(\bigcup_{n=1}^{3} A_n\right) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_2 \cap A_3) \\ &= 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{216} = \frac{91}{216} \approx 0.421. \end{aligned}$$

2.111

a) On any particular toss of the die, we are interested only in whether the result is a six or not, which we denote s and f, respectively. The game stops at the nth toss of the die if and only if the first $n-1$ tosses are not sixes and the nth toss is a six, which we can represent as the n-tuple, $(f, f, \ldots, f, s)$. Hence, a sample space is

$$\Omega = \Big\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s) : n \in \mathcal{N}\Big\} \cup \{(f, f, \ldots)\},$$

where $(f, f, \ldots)$ represents the outcome that a six is never tossed (i.e., the game never stops).

b) Because the die is balanced, when it is tossed n times, the possible outcomes are equally likely. From the hint, there are 6^n possible outcomes of which 5^{n-1} have the property that the first $n-1$ tosses are not sixes and the nth toss is a six. Hence,

$$P\big(\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s)\}\big) = \frac{5^{n-1}}{6^n} = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}.$$

As we discover in part (c), the game must eventually stop. This implies that $P\big(\{(f, f, \ldots)\}\big) = 0$.

c) Let E denote the event that the game eventually stops. Then, from Proposition 2.2 on page 42 and part (b), we have

$$P(E) = \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{n-1 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6} \cdot \frac{1}{1-5/6} = 1.$$

d) Let T denote the event that Tom wins. As Tom goes first, event T occurs if and only if the first six occurs on trial 1 or 4 or 7 or ..., that is, on trial $3n-2$ for some $n \in \mathcal{N}$. Hence, from Proposition 2.2

and part (b), we have

$$P(T) = \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{3n-3 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{3n-3} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\left(\frac{5}{6}\right)^{3}\right)^{n-1}$$

$$= \frac{1}{6} \cdot \frac{1}{1-(5/6)^3} = \frac{36}{91} \approx 0.396.$$

Let D denote the event that Dick wins. As Dick goes second, event D occurs if and only if the first six occurs on trial 2 or 5 or 8 or ..., that is, on trial $3n-1$ for some $n \in \mathcal{N}$. Hence, from Proposition 2.2 and part (b), we have

$$P(D) = \sum_{n=1}^{\infty} P\big(\{(\underbrace{f, f, \ldots, f}_{3n-2 \text{ times}}, s)\}\big) = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{3n-2} = \frac{1}{6} \cdot \frac{5}{6}\sum_{n=1}^{\infty}\left(\left(\frac{5}{6}\right)^{3}\right)^{n-1}$$

$$= \frac{5}{36} \cdot \frac{1}{1-(5/6)^3} = \frac{30}{91} \approx 0.330.$$

Let H denote the event that Harry wins. Then, from part (c), the complementation rule, and the additivity axiom, we deduce that

$$P(H) = 1 - P(H^c) = 1 - P(T \cup D) = 1 - P(T) - P(D) = 1 - \frac{36}{91} - \frac{30}{91} = \frac{25}{91} \approx 0.275.$$